MASSACHUSETTS INSTITUTE OF TECHNOLOGY
HAYSTACK OBSERVATORY
WESTFORD, MASSACHUSETTS 01886
21 November 2001

Telephone: 978-692-4764
Fax: 781-981-0590

To: LOFAR Group
From: Shep Doeleman
Subject: Simulating LOFAR Station Beams

This memo lays the groundwork for describing complex valued LOFAR station beams. We assume that a station will comprise many individual receptors at arbitrary positions and that we will want to phase the station in real time to track a fixed RA and DEC. Sections 1 and 2 deal with simple 1-D and 2-D receptor configurations and section 3 generalizes the problem of beam forming for a 3-D receptor pattern. Section 4 gives the relevant expressions for a LOFAR station beam both as a function of (Az,El) and of (RA,DEC). Code to model LOFAR station beams is available that can accept lists of antenna positions, create regular grids of receptors, or generate a random receptor pattern. Crosstalk between receptors is not handled by this code and the analysis below assumes that crosstalk effects are negligible.

1 1-D Case

We first take the one dimensional case with all antennas constrained to lie on the x-axis. All antenna positions will be measured relative to $x = 0$ and the zenith angle (θ) will be measured from the z-axis towards the positive x-axis where we assume the usual right hand Cartesian coordinate system (see Fig. 1). For radiation arriving at the station from a direction θ, the geometrical phase offset for the jth receptor in the array will be :

$$\chi_j = -2\pi x_j \frac{\nu}{c} \sin\theta \tag{1}$$

where x_j is the station position, ν is the observing frequency, c is the speed of light, and χ_j is the phase offset. To electronically steer the station beam to a direction $\theta_\circ$, phase offsets are introduced for each receptor that exactly cancel the geometrical phase offset for that specific direction. These steering phase offsets are

$$\eta_j = 2\pi x_j \frac{\nu}{c} \sin\theta_\circ \tag{2}$$

so that for $\theta = \theta_\circ$, $\eta_j + \chi_j = 0$ and all receptors in the array can be added coherently. We can then write an expression for the phased beam by adding the responses of all the receptors (N= number of receptors)

$$B(\theta) = \sum_{j=1}^{N} \exp\left(i(\chi_j + \eta_j)\right) = \sum_{j=1}^{N} \exp\left(-2\pi i x_j \frac{\nu}{c}(\sin\theta - \sin\theta_\circ)\right) \tag{3}$$

An example in Fig 2 shows the effects of adding receptors to an array. Panels is this figure show amplitude and phase characteristics for arrays with 2,3,4,5, and 6 receptors. Here we define the beam as the amplitude and phase response of the vector sum of receptor signals as a function of angle on the sky.

2 2-D Case

In the two dimensional case, we will use polar coordinates to represent the receptor placement (r_j, ϕ_j). Coordinates θ and ϕ will be used to represent position on the sky with θ measured from the z-axis and ϕ measured from the x-axis towards the positive y-axis (in the $\theta = \pi/2$ plane). Note that the ϕ coordinate is used both for receptor position *and* for sky position (see Fig. 3). The geometrical phase offset for each receptor from a direction (θ, ϕ) will be

$$\chi_j = -2\pi r_j \frac{\nu}{c} \sin\theta \cos(\phi_j - \phi) \tag{4}$$

The expression for the 2-D beam is then

$$B(\theta, \phi) = \sum_{j=1}^{N} \exp\left(-2\pi i r_j \frac{\nu}{c}(\sin\theta\cos(\phi_j - \phi) - \sin\theta_\circ \cos(\phi_j - \phi_\circ))\right) \tag{5}$$

Generally, each receptor will have its own beam pattern and the expression in Eq. 5 must be multiplied by this function (of (θ, ϕ)) to get the full station beam. Eq. 5 also neglects the curvature of the earth within a station since this only introduces positional offsets of $\pm$3mm or $\sim \frac{1}{400}$ of a wavelength at the highest LOFAR frequency.

3 Vector Notation

The preceding sections use notation that is specific to a given coordinate system, but it is possible and desirable to express the phased station beam in a generalized vector notation. Let us assume that the receptors reside in a three dimensional space and their positions are $\vec{d_j}$, and $\vec{d_\circ}$ will be a reference position in the array plane. A unit vector pointing to a position on the sky will be $\hat{s}$ and $\hat{s}_\circ$ will be the point on the sky for which we wish to phase the station. Let $k = 2\pi\frac{\nu}{c}$ and A_j will be a positive, real weight for the jth receptor. The general beam pattern can then be expressed as

$$B(\hat{s}) = \sum_{j=1}^{N} A_j \exp\left(-ik(\vec{d_j} - \vec{d_\circ}) \cdot (\hat{s} - \hat{s}_\circ)\right) \tag{6}$$

To ensure that the phase across the main lobe of the beam has zero slope, one sets the reference station position (sometimes referred to as the "zero-pad position") to the weighted mean position of all receptors in the array

$$\vec{d_\circ} = \frac{\sum_{j=1}^{N} A_j \vec{d_j}}{\sum_{j=1}^{N} A_j} \tag{7}$$

This relation can be obtained by differentiating the expression for beam phase $(\mathrm{Arg}(B(\vec{s})))$ and setting the result equal to zero:

$$\mathrm{Re}(B(\hat{s})) = \sum_{j=1}^{N} A_j \cos\left(k(\vec{d_j} - \vec{d_\circ}) \cdot (\hat{s} - \hat{s}_\circ)\right) \tag{8}$$

$$\mathrm{Im}(B(\hat{s})) = -\sum_{j=1}^{N} A_j \sin\left(k(\vec{d_j} - \vec{d_\circ}) \cdot (\hat{s} - \hat{s}_\circ)\right) \tag{9}$$

$$\mathrm{Arg}(B(\hat{s})) = \arctan\left(\mathrm{Im}(B(\hat{s}))/\mathrm{Re}(B(\hat{s}))\right) \tag{10}$$

4 The LOFAR case

For the LOFAR simulator, we want to specify the station beam given the following: the Hour Angle ($\mathcal{H}$) and declination (δ) of the target source, the latitude ($\mathcal{L}$) of the station, and 3-D positions of the station receptors relative to the fiducial station center. We start with a right hand coordinate frame in which $\hat{x}$ is North and $-\hat{y}$ is East. Elevation ($\mathcal{E}$) and Azimuth ($\mathcal{A}$) will initially be used to describe positions on the sky where Azimuth is measured East of North (see Figure 4). Using Eq. 7, we can write the beam shape as

$$\begin{aligned} B(\mathcal{A},\mathcal{E}) \;=\; \sum_{j=1}^{N} A_j \exp\big(-ik\,\{ & (d_{jx}-d_{\circ x})(\cos\mathcal{E}\cos\mathcal{A}-\cos\mathcal{E}_\circ\cos\mathcal{A}_\circ) \\ & + (d_{jy}-d_{\circ y})(-\cos\mathcal{E}\sin\mathcal{A}+\cos\mathcal{E}_\circ\sin\mathcal{A}_\circ) \\ & + (d_{jz}-d_{\circ z})\;(\sin\mathcal{E}-\sin\mathcal{E}_\circ)\}\big) \end{aligned} \tag{11}$$

where $(\mathcal{A}_\circ, \mathcal{E}_\circ)$ is the point on the sky for which the array is phased. Figure 5 shows a sample 2-D beam formed using 64 randomly placed receptors within a 50 meter radius station. This expression can also be used to investigate the beam of a HIFAR element consisting of a grid of dipoles. Figure 6 shows the beam resulting from a 4x4 dipole array with spacings of $3\lambda/4$.

To convert to Hour Angle and declination, we use the relations:

$$\begin{aligned} \sin\mathcal{E} &= \sin\mathcal{L}\sin\delta+\cos\mathcal{L}\cos\delta\cos\mathcal{H} \\ \cos\mathcal{E}\cos\mathcal{A} &= \cos\mathcal{L}\sin\delta-\sin\mathcal{L}\cos\delta\cos\mathcal{H} \\ \cos\mathcal{E}\sin\mathcal{A} &= -\cos\delta\sin\mathcal{H} \end{aligned} \tag{12}$$

and rewrite the beam:

$$\begin{aligned} & B(\mathcal{H},\delta) = \\ & \sum_{j=1}^{N} A_j \exp\big(-ik\,\{(d_{jx}-d_{\circ x})(\cos\mathcal{L}(\sin\delta-\sin\delta_\circ)-\sin\mathcal{L}(\cos\delta\cos\mathcal{H}-\cos\delta_\circ\cos\mathcal{H}_\circ) \\ & \qquad + (d_{jy}-d_{\circ y})(\cos\delta\sin\mathcal{H}-\cos\delta_\circ\sin\mathcal{H}_\circ) \\ & \qquad + (d_{jz}-d_{\circ z})\;(\sin\mathcal{L}(\sin\delta-\sin\delta_\circ)-\cos\mathcal{L}(\cos\delta\cos\mathcal{H}-\cos\delta_\circ\cos\mathcal{H}_\circ)\}\big) \end{aligned} \tag{13}$$

This expression gives the instantaneous complex station beam for a 3-D grid of receptors as a function of $\mathcal{H}$ and δ. This effectively yields the beam as a function of RA and Dec since $\mathcal{H}$ = GAST + (East Longitude) - RA, where GAST is the Greenwich Apparent Sidereal Time. Placing the beam in this form facilitates comparison between all array station beams in a common region of the sky.

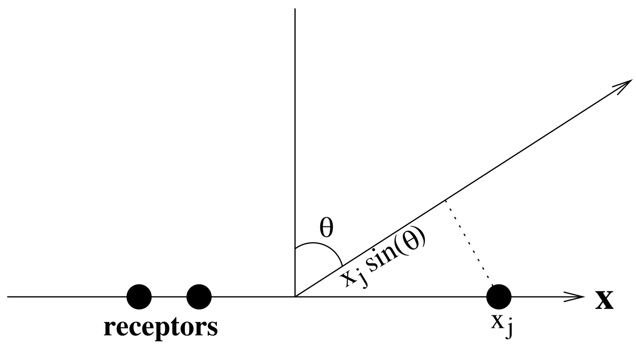

Figure 1: Figure for the 1-D case with co-linear receptors.

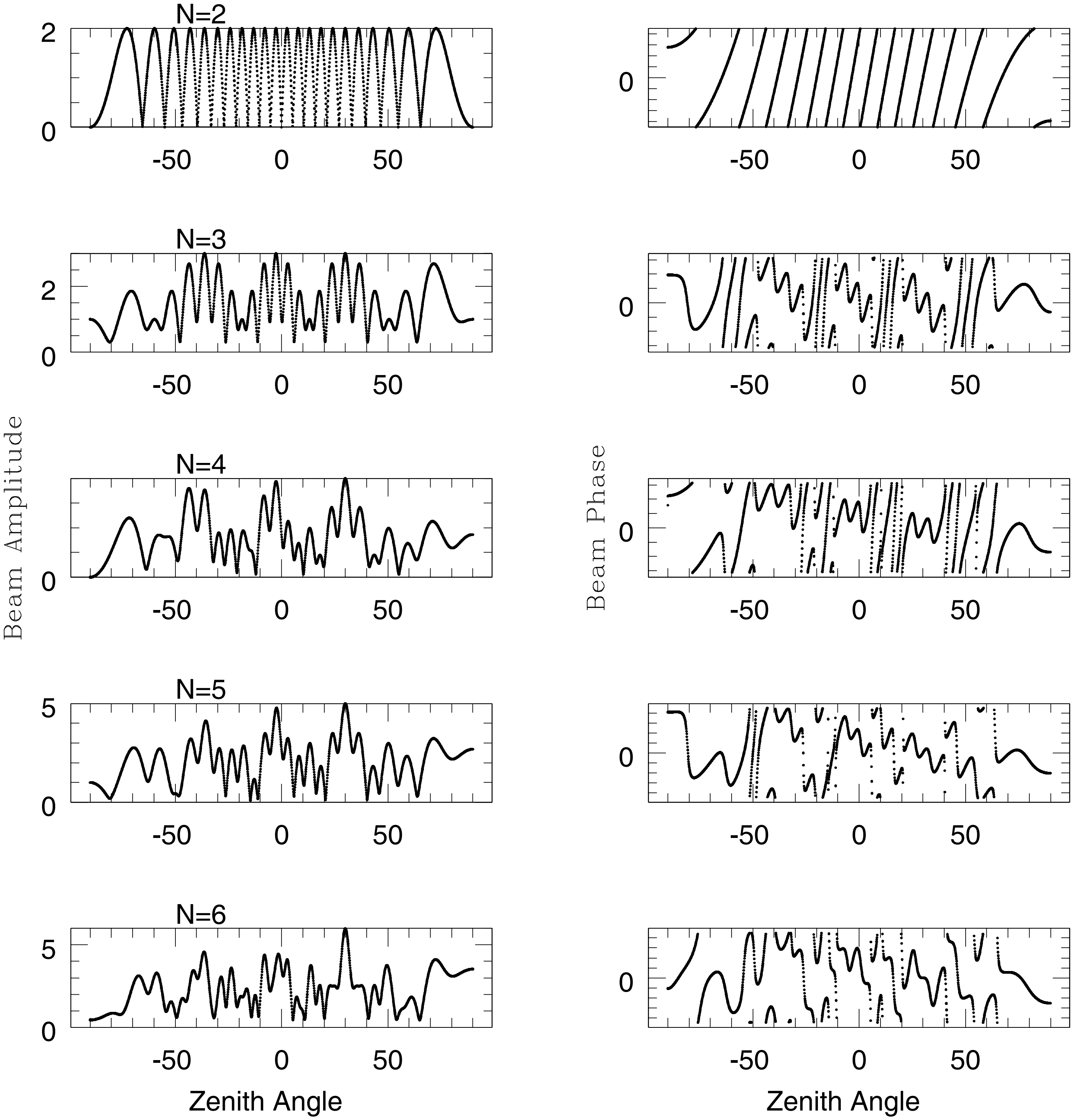

Figure 2: Sample beams for the 1-D case. Beam amplitude (left) and phase (right) responses are shown for 5 different arrays. Top panel has two receptors and an additional receptor is added to the array in each subsequent pair of plots. The phase reference point of the array is the weighted mean of all six final receptor positions so in the last pair of plots the phase slope across the main beam (at $\theta = 30$) is zero. Antenna positions are (in units of $x\frac{\nu}{c}$: -5.5, 5.5, 3.67, 2, 5.714, 10. The phase reference point of the array is at $x = 3.564$.

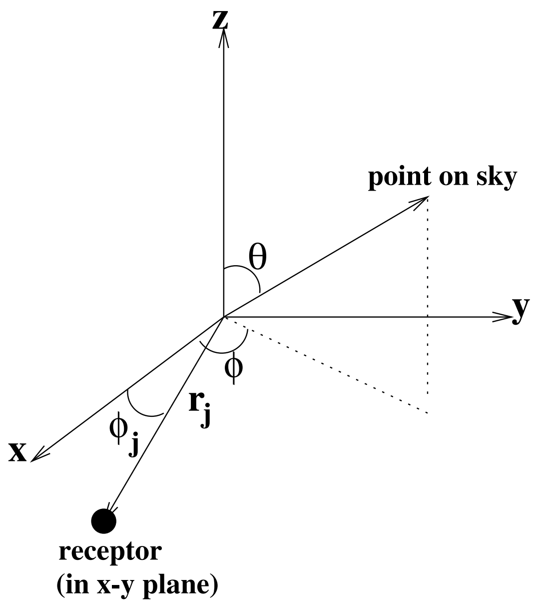

Figure 3: Figure for 2-D case of Equations 4 and 5 where receptors are confined to x,y plane and polar coordinates are used.

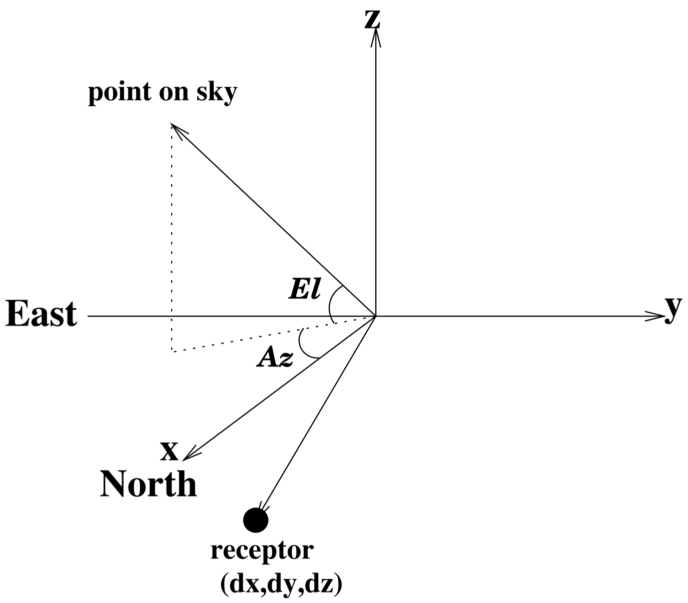

Figure 4: Figure for the LOFAR case of Equations 11 and 13 where receptors can be distributed in 3 dimensions relative to the station center. Note the directions of the x axis (North) and y axis (East) which are important for conversion to Hour Angle and declination. Using Hour Angle and Declination allows easy comparison of beams between different stations.

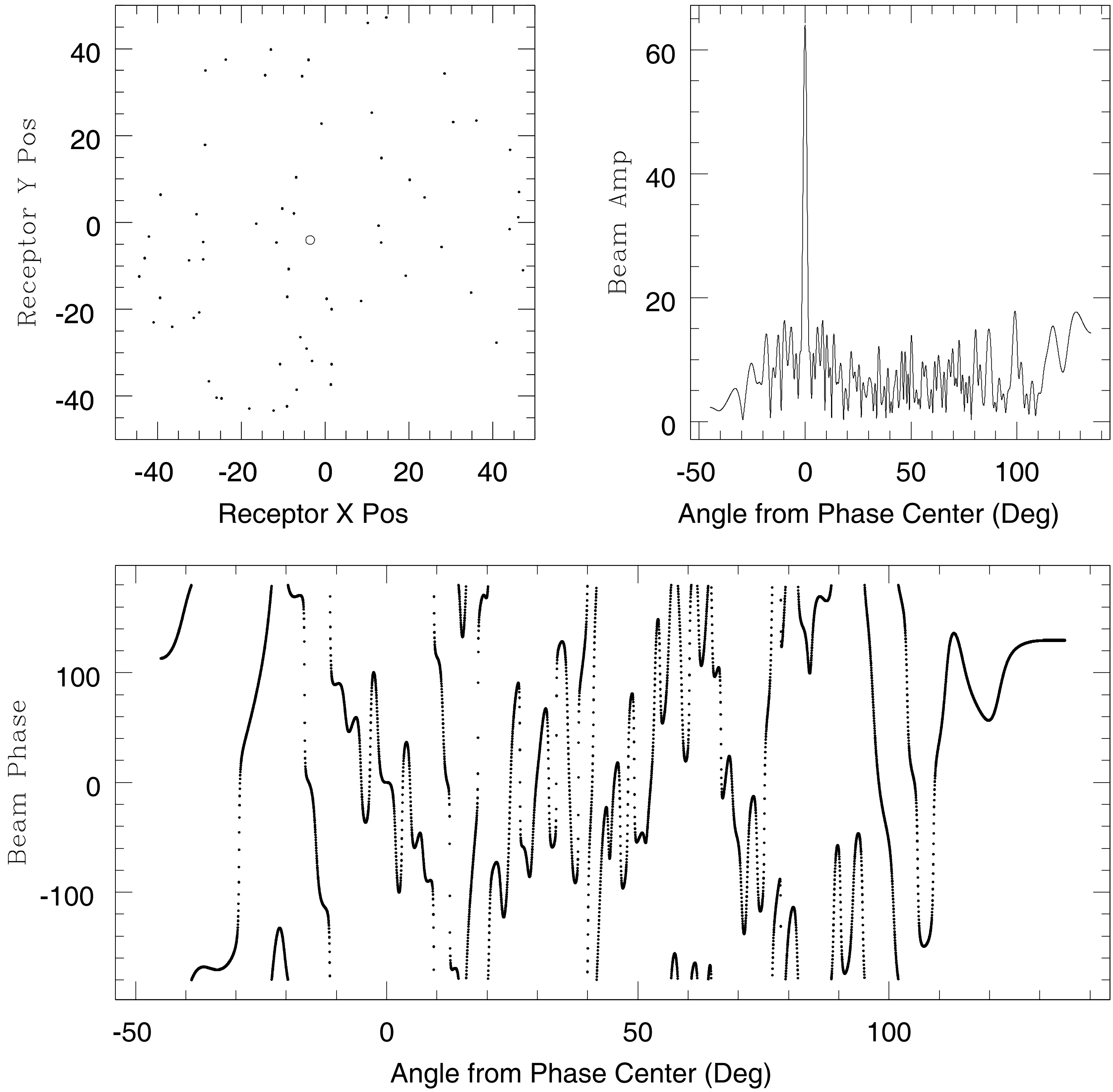

Figure 5: Sample LOFAR station beam composed of 64 randomly placed stations within a 50 meter radius. The observing frequency is 150MHz. Upper left panel shows the station placement with the weighted mean receptor position shown by the open circle. Resulting beam is computed from Eq. 11 with phase center at Az=0 and El=45 degrees. The beam cut shown in the upper panel is through the phase center in the North direction with degrees along the x-axis. Phase in the lower plot is along the same cut. Note the zero slope in phase for the main beam lobe.

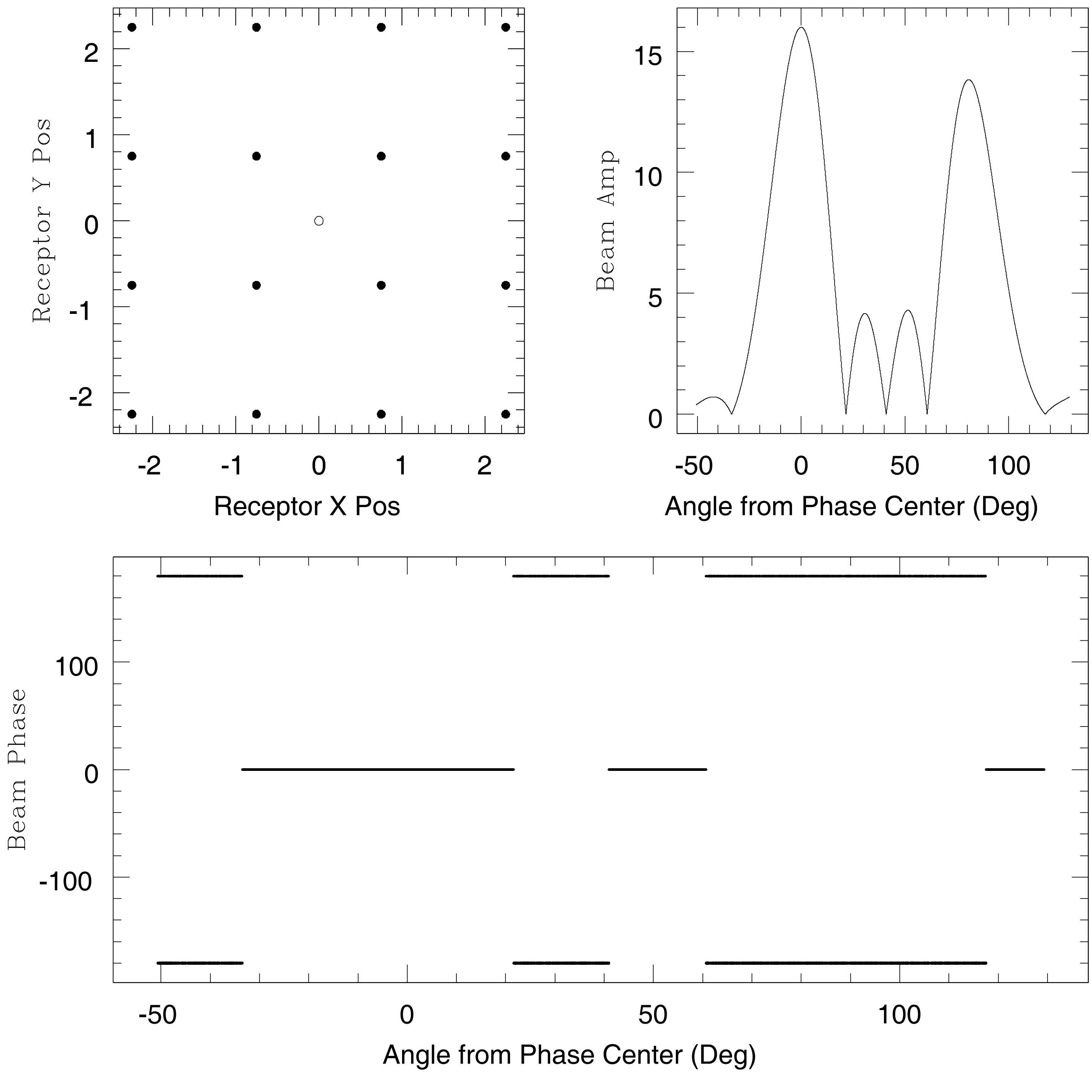

Figure 6: Sample beam from a HIFAR 4x4 regular grid of dipoles with 1.5m spacing at an observing frequency of 150MHz ($\lambda = 2$m). The phase center on the sky is Az=30, El=45 degrees. The beam amplitude and phase are shown for a cut through the beam center along a direction 35 degrees West of North. Note that for all regular receptor patterns with infinite phasing resolution like this one, the phase for each lobe will always be 0 or π.