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To: Holographers

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Subject: Holography resolution functions

This pedagogical note was motivated by Alan Rogers's 19 March 1992 memo on the panel structure that is unresolved by 91×91 holography. The purpose is to present the analytical forms for the resolution functions and their corresponding frequency response functions. Some approximations are made (*e.g.*, I use continuous Fourier transforms here, whereas in practice FFT's are used), but the effects on the results are negligible. All integrals extend from $-\infty$ to $+\infty$.

Before transformation to the map plane, the complex beam data are first gridded onto the rectangular (u, v) grid to form the data set $B(u, v)$, and then a window function $W(u, v)$ is applied to the data. u and v are the direction cosines $u = \sin\theta\cos\phi$ and $v = \sin\theta\sin\phi$, where θ and ϕ are the polar and azimuthal angles about the center of the beam.

The process of transforming to the map plane may be expressed as

$$M(x,y) \;=\; \iint B(u,v)\, W(u,v)\, e^{ik(ux+vy)}\, du\, dv \tag{1}$$

where $M(x, y)$ is the complex map in the (x, y) aperture plane, and $k = 2\pi/\lambda$. In the usual manner, eq. (1) may also be written as the 2D convolution of two functions:

$$M(x,y) \;=\; \frac{k^2}{4\pi^2} \iint M_o(x-\xi, y-\eta)\, R(\xi,\eta)\, d\xi\, d\eta \tag{2}$$

where $M_o(x, y)$ is the map obtained from eq. (1) with $W(u, v) = 1$ everywhere (*i.e.*, it is the map with the highest possible resolution, for which the beam pattern has been measured over the full sky), and $R(x, y)$ is the resolution function given by

$$R(x,y) \;=\; \iint W(u,v)\, e^{ik(ux+vy)}\, du\, dv \tag{3}$$

We have used two different window functions in the analysis of holography data. The original *square* window has the form

$$\begin{aligned} W_s(u,v) &= 1 \text{ for } |u| \le u_{max},\ |v| \le u_{max} \\ &= 0 \text{ elsewhere} \end{aligned}$$

More recently the standard has become the *circular* function

$$\begin{aligned} W_c(u,v) &= 1 \text{ for } \sqrt{u^2+v^2} \le u_{max} \\ &= 0 \text{ elsewhere} \end{aligned}$$

For a $(2N+1)\times(2N+1)$ map, u_{max} is $N\Delta u$, where Δu is the spacing between grid points in the (u,v) plane. All Haystack maps have had $\Delta u = \sin 0.032^\circ = 0.00056$. For a 91×91 map, $u_{max} = 0.0251$.

The corresponding resolution functions in the map plane, obtained from eq. (3) and normalized to unity at the origin, are

$$R_s(x,y) = \frac{\sin kxu_{max}}{kxu_{max}} \frac{\sin kyu_{max}}{kyu_{max}} \tag{4}$$

$$R_c(x,y) = 2\,\frac{J_1(kru_{max})}{kru_{max}} \tag{5}$$

where $r = \sqrt{x^2+y^2}$, and J_1 is the Bessel function of the first kind and first order. Cross-sections through these functions for a 91×91 map are plotted in Figure 1. The resolution function represents the "impulse response" of the system, in the sense that a delta-function spike in the high-resolution map $M_o(x,y)$ gets smeared out by the windowing process to look like $R(x,y)$.

A complementary way to look at the effect of the window function is to compute the frequency response function $S(k_x,k_y)$, which is the inverse Fourier transform of the resolution function:

$$S(k_x,k_y) = \frac{k^2}{4\pi^2}\iint R(x,y)\, e^{-i(k_x x+k_y y)}\, dx\, dy \tag{6}$$

The spectral content of the map is simply $S(k_x,k_y)$ times the spectral content of the unwindowed map $M_o(x,y)$. By eq. (3) [or you can do the integrals, as a check of eqs. (4) and (5)],

$$S(k_x,k_y) = W(k_x/k, k_y/k) \tag{7}$$

For instance, the frequency response function for the circular window is

$$S(k_x,k_y) = 1 \text{ for } \sqrt{k_x^2+k_y^2} \le ku_{max} \tag{8}$$

$$= 0 \text{ elsewhere} \tag{9}$$

For $u_{max} = 0.0251$ and $\lambda = 2.54$ cm, $ku_{max} = 2\pi/$ (101 cm). Hence, a 91×91 map produced with a circular window will exhibit unattenuated all power on wavelength scales > 100 cm across the aperture, and will have no power on shorter scales.

One caveat: Strictly speaking, the resolution function and frequency response function do not apply to the map *phase*, since the phase is a nonlinear function of the map $M(x,y)$, and Fourier transformation is a linear process. In the case of small ($\ll 1$ radian) map phases and large-scale-only ($> \lambda/u_{max}$) amplitude variations, however, the resolution function may be applied directly to the map phases. For 91×91 maps, these two conditions hold reasonably well over most of the Haystack aperture plane, except at the edge of the dish and near the quadrapod legs and subreflector.

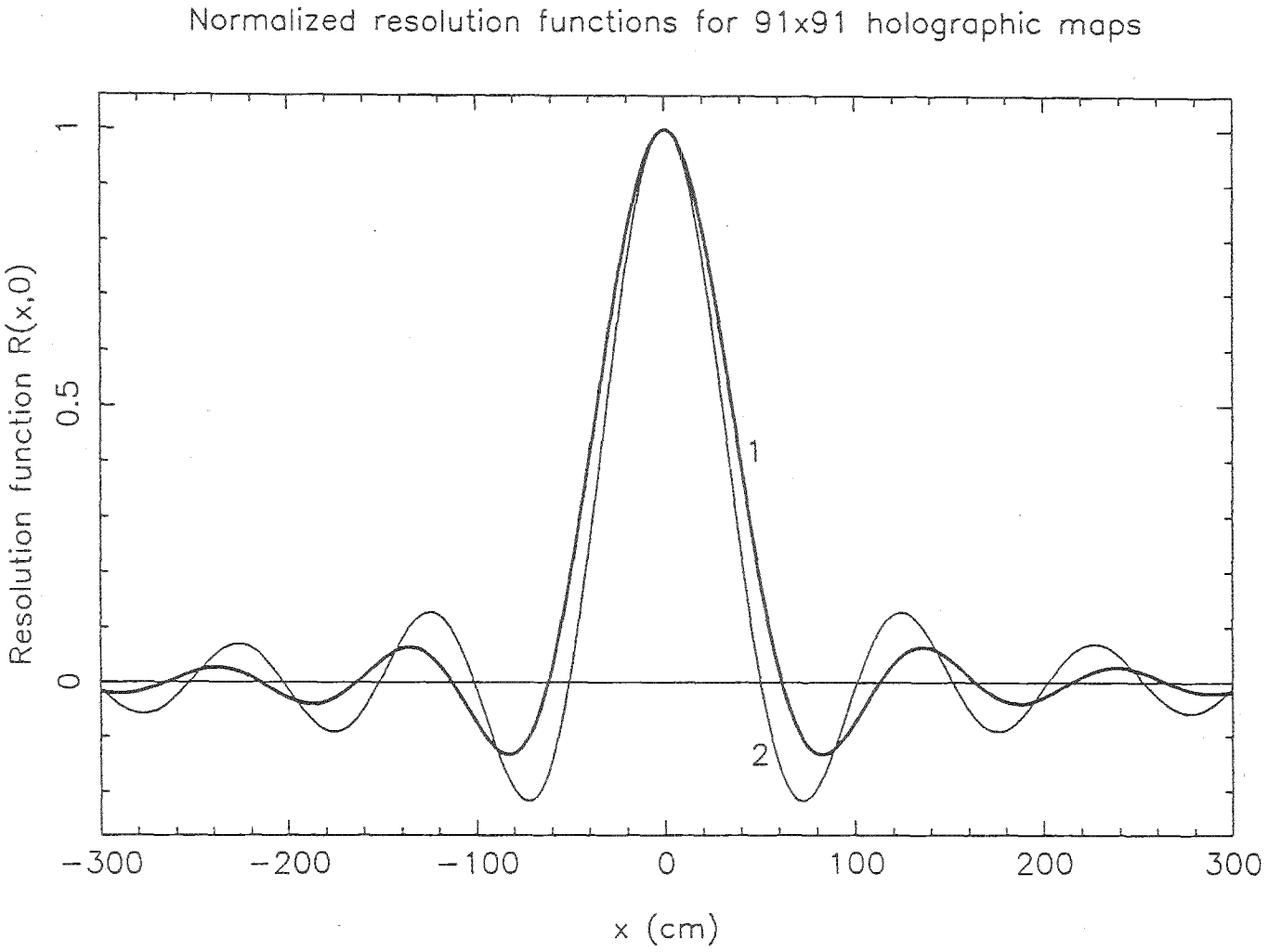


Figure 1. Cross-section through origin, parallel to x-axis, of normalized resolution functions for 91×91 holography maps, with $u_{max} = 0.251$ and $\lambda = 2.54$ cm. (1) Circular window resolution function $R_c(x, y)$, which is circularly symmetric. (2) Square window resolution function $R_s(x, y)$.