Mark IV Memo #276

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To : VLBI Group

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Title : **Deformation of a Tape Subject to Uniform Negative Pressure: No-Contact Case.**

Introduction

The goal of this memorandum is to present the mechanics of the tape deformation underlying the operation of a flat-head/tape interface. As a first order of approximation, a simplified system is analyzed; where a uniform negative pressure p^* is assumed to be acting on the tape, only over the head region; and where the analysis is carried out for those values of p^* that do not cause contact with the head over the central region. Figure 1 shows the generic geometry that is considered.

With the assumptions mentioned above, the problem can be modeled with a constant coefficient, ordinary differential equation, and a closed form solution can be obtained. This solution establishes the relation between the tape displacements and the wrap angle, tape tension, bending rigidity, head-length and external pressure. The solution is given in normalized coordinates, which are defined in the next section.

The following pages will show that the ratio of the head-length Δ to the bending length b of the tape plays a critical role in this problem; the behavior of the tape deformation reverses itself from *increasing cupping of the tape with increasing tension* to *flattening with increasing tension* at $(\Delta/b)_{cr}$.

In the analysis presented below the pressure values are considered in the range $0 \leq p^* < p_{cr}^*$. The lower limit corresponds to the case where the tape is wrapped over a flat-head under tension, but without any negative pressure acting downward; and the upper limit is the pressure which causes the first contact. It is shown that p_{cr}^* value is a function of the other parameters of the analysis.

The case where the tape starts to contact the head (i.e., $p^* > p_{cr}^*$) requires the analysis of a non-linear "contact problem." That will be the subject of a different memo.

The Governing Equation

If the effects of tape width are neglected and if only the pressure p^* values which cause positive deflections over the head-region (i.e., $w^* > 0$, $0 < x^* < \Delta$) are considered, then the tape deflections can be modeled by the Euler-Bernoulli beam equation,

$$D\frac{d^4w^*}{dx^{*4}} - T\frac{d^2w^*}{dx^{*2}} = -p^*\left(H(x^*) - H(x^* - \Delta)\right) \tag{1}$$

where,

w^* is the tape displacement measured from the head surface,

x^* is the coordinate axis along the tape with the origin located on the left corner of the head,

p^* is the uniform pressure acting on the tape in the negative w^* direction as shown in Figure 1,

Δ is the length of the head in the running direction,

D is the bending rigidity of the tape given by $D = Ec^3/12(1-\nu^2)$,

T is the tension per unit width,

E is the modulus of elasticity,

c is the thickness,

ν is the Poisson's ratio, and

H is the Heaviside step function.

The wrap angles θ^*_L and θ^*_R are used set the tape wrap over the head. Note that in this approach the p^* values that cause positive deflections over the head are not known apriori. But once a general solution is found then the appropriate range for p^* can be determined. For the configuration shown in Figure 1 the boundary conditions are as follows:

a. On the left and right sides of the head, the wrap angles are as shown in the figure and the tape displacements are bounded. This is represented as,

$$x^* \to -\infty: \quad \frac{dw^*}{dx^*} = \theta^*_L, \quad \text{and} \quad w^* < \infty \qquad\qquad x^* \to \infty: \quad \frac{dw^*}{dx^*} = \theta^*_R, \quad \text{and} \quad w^* < \infty \tag{2a-d}$$

b. On the edges of the head, the tape displacement is zero and the slope and the curvature of the tape are continuous. These are represented as,

$$\begin{aligned}
x^* = 0:\quad & w^{*(1)} = w^{*(2)} = 0, & x^* = \Delta:\quad & w^{*(2)} = w^{*(3)} = 0,\\
& \frac{dw^{*(1)}}{dx^*} = \frac{dw^{*(2)}}{dx^*}, & & \frac{dw^{*(2)}}{dx^*} = \frac{dw^{*(3)}}{dx^*},\\
& \frac{d^2w^{*(1)}}{dx^{*2}} = \frac{d^2w^{*(2)}}{dx^{*2}}, & & \frac{d^2w^{*(1)}}{dx^{*2}} = \frac{d^2w^{*(2)}}{dx^{*2}}.
\end{aligned} \tag{3a-h}$$

Normalization of the Governing Equations

The following normalization is used for the beam equation and the boundary conditions,

$$x = \frac{x^*}{\Delta}, \quad w = \frac{w^*}{c}, \quad p = \frac{p^*\Delta^2}{Tc}, \quad \theta = \frac{\Delta\theta^*}{c} \tag{4}$$

Then equation (1) becomes,

$$\frac{d^4 w}{dx^4} - \left(\frac{\Delta}{b}\right)^2 \frac{d^2 w}{dx^2} = -p\left(\frac{\Delta}{b}\right)^2 \left(H(x) - H(x-1)\right), \tag{5}$$

where the parameter $b = (D/T)^{1/2}$ is the *bending length* of the tape. Note that the parameter Δ/b can also be expressed as,

$$\frac{\Delta}{b} = 2\sqrt{3(1-\nu^2)}\,\frac{\Delta T^{1/2}}{E^{1/2}c^{3/2}} \tag{6}$$

Thus when all other parameters are constant, increasing Δ/b values can be thought as increasing tape tension T.

Note that in this analysis,

- w is the dependent variable,
- x is the independent variable, and
- p, (Δ/b), θ_L, *and* θ_R are the parameters of the analysis.

General Solution

Equation (5) has the following piecewise solution,

$$w(x) = \begin{cases} w^{(1)} = A_1 + B_1 x + C_1 e^{-\Delta x/b} + D_1 e^{\Delta x/b}, & \text{for } x \le 0, \\ w^{(2)} = A_2 + B_2 x + C_2 e^{-\Delta x/b} + D_2 e^{\Delta x/b} + \frac{1}{2}\mathrm{p}\left(\frac{\Delta}{\mathrm{b}}\right)^2 \mathrm{x}^2, & \text{for } 0 \le x \le 1, \\ w^{(3)} = A_3 + B_3 x + C_3 e^{-\Delta x/b} + D_3 e^{\Delta x/b}, & \text{for } x \ge 0, \end{cases} \tag{7}$$

The unknown coefficients are obtained from the boundary conditions.

Solution for the $\theta_L = \theta_R = \theta$ Case

The solution for the general case of $\theta_L \neq \theta_R$ is straightforward, but has a more complicated composition than $\theta_L = \theta_R$. It is easier to interpret the results of the equal wrap angle case; in this memo, the case of $\theta_L = \theta_R = \theta$ is considered. After the necessary substitutions, the solution becomes,

$$w_1(x) = \theta x - \frac{1}{4}(\frac{b}{\Delta})\left[p\left((1-2\frac{b}{\Delta}) + (1+2\frac{b}{\Delta})e^{-\Delta/b} \right) + 2(1+e^{-\Delta/b}) \right]\left(e^{\Delta x/b} - 1 \right)$$

$$w_2(x) = \frac{1}{2}p(x^2 - x) - \frac{1}{4}(\frac{b}{\Delta})\left[p(1+2\frac{b}{\Delta}) + 2\theta \right]\left[(e^{-\Delta x/b} - 1) + (e^{\Delta(x-1)/b} - e^{-\Delta/b}) \right]$$

$$w_3(x) = \theta(1-x) - \frac{1}{4}(\frac{b}{\Delta})\left[p\left((1-2\frac{b}{\Delta}) + (1+2\frac{b}{\Delta})e^{-\Delta/b} \right) + 2(1+e^{-\Delta/b}) \right]\left(e^{\Delta(1-x)/b} - 1 \right)$$

(8a-c)

Deflection Behavior of the Tape

Using the solution given by equation (8) we can determine:

a) The general deflection shape of the tape and,
b) The middle point deflection of the tape, $w_m = w^{(2)}(1/2)$,

as a function of the parameters; (Δ/b) ratio, wrap angle θ and pressure p.

First, however, we should determine the critical pressure p_{cr} above which tape deflections become negative over the head region. The middle point of the tape will make the first contact with the head, under the uniform pull-down pressure p, with symmetrical wrap. The mid-point deflection of the head is obtained from equation (8b) as,

$$w_m = w^{(2)}(\frac{1}{2}) = \frac{1}{4}\left[(\frac{b}{\Delta})\left(p(1+2(\frac{b}{\Delta})) + 2\theta \right)\left(e^{-0.5\Delta/b} - 1 \right)^2 - \frac{1}{2}p \right] \quad (9)$$

Then the critical pressure p_c can be obtained by setting $w_m = 0$,

$$p_{cr} = \frac{4(\frac{b}{\Delta})\theta\left(e^{-0.5\Delta/b} - 1 \right)^2}{1 - 2(\frac{b}{\Delta})\left(1 + 2(\frac{b}{\Delta}) \right)\left(e^{-0.5\Delta/b} - 1 \right)^2} \quad (10)$$

The critical pressure p_{cr} varies linearly with the wrap angle θ, as can be seen by inspection of equation (10). But, a non-linear relation exists between the parameter Δ/b and p_{cr}; this relationship is such that limit of p_{cr} as $\Delta/b \to \infty$ is zero and the limit of p_{cr} as $\Delta/b \to 0$ is infinity[1] as shown in Figure 2. Based on this figure we see that the lower wrap angles have a lower critical pressure; in other words if the wrap angle is low the pressure to cause initial contact is also low.

[1] However, we should be careful to note that the case Δ/b=0 has a different solution than equation (7). In essence, the case of Δ/b = 0 corresponds to the same problem without tension. The solution to that problem is given in the Appendix.

The mid-point displacement of the tape w_m, given in equation (9), varies linearly with the external pressure p and the wrap angle θ. But its relation to the parameter Δ/b is non-linear. The w_m-*vs*-Δ/b variation for different external pressure and wrap angle values are given in Figure 3. This plot shows that for any given p-θ pair w_m-vs-Δ/b curves have a peak at a specific value of Δ/b indicated by $(\Delta/b)_{cr}$. To the left of this peak the mid-point displacement increases with increasing Δ/b, and to the right they decrease.

The effect of this interesting observation, on the overall tape displacement is shown in Figure 4. Part a) of this figure shows the case of $\Delta/b < (\Delta/b)_{cr}$, and part b) shows the case of $\Delta/b > (\Delta/b)_{cr}$. These plots show that increasing the value of the parameter Δ/b, when it is less than critical, "cupping" of the tape becomes more pronounced. In contrast, increasing Δ/b, when $\Delta/b > (\Delta/b)_{cr}$, causes the tape become flatter.

The critical value of the parameter Δ/b can be obtained by solving the following non-linear equation,

$$\frac{-4-(\frac{\Delta}{b})_{cr}\left(3+(\frac{\Delta}{b})_{cr}\right)+\left(4+(\frac{\Delta}{b})_{cr}\right)e^{\frac{1}{2}(\frac{\Delta}{b})_{cr}}}{2(\frac{\Delta}{b})_{cr}\left(1+(\frac{\Delta}{b})_{cr}-e^{\frac{1}{2}(\frac{\Delta}{b})_{cr}}\right)}-\frac{\theta}{p}=0 \qquad (11)$$

The solution of this equation is plotted on Figure 3. It is interesting to note that the largest value of $(\Delta/b)_{cr}$ (=2.51286) occurs when p = 0. In other words, when there is no external loading on the tape, the change in displacement behavior of the tape occurs when Δ/b = 2.51286; and this value is independent of the wrap angle θ. In contrast, when there is external load acting downward on the tape, the critical value of the parameter Δ/b occurs sooner than 2.51286; and in this case the critical value is dependent on the wrap angle.

The Parameters for the VLBI Tape Recorders

Typical parameters for the VLBI tape recorders are as follows:

Dimensional Variables	
Elastic modulus, E	4 GPa
Thickness, c	15.2 μm
Poisson's ratio, ν	0.3

Bending rigidity, D	1.286×10^{-6} Nm
Tape tension, T	43.5 - 130.4 N/m (5"-15" H_20)
Bending length, b	172 - 99.3 μm
Wrap angle, θ^*	5^o
Head length, Δ	1.5 mm (Proposed VLBI Flat Head)
Normalized Variables	
Δ/b	**8.7 - 15.1**
θ	**8.6**
p_{cr}	**5.32 - 2.68 (From eqn. (10))**
Critical Pressure (Dimensional)	
p^*_{cr}	**1564.7 - 2361 Pa**

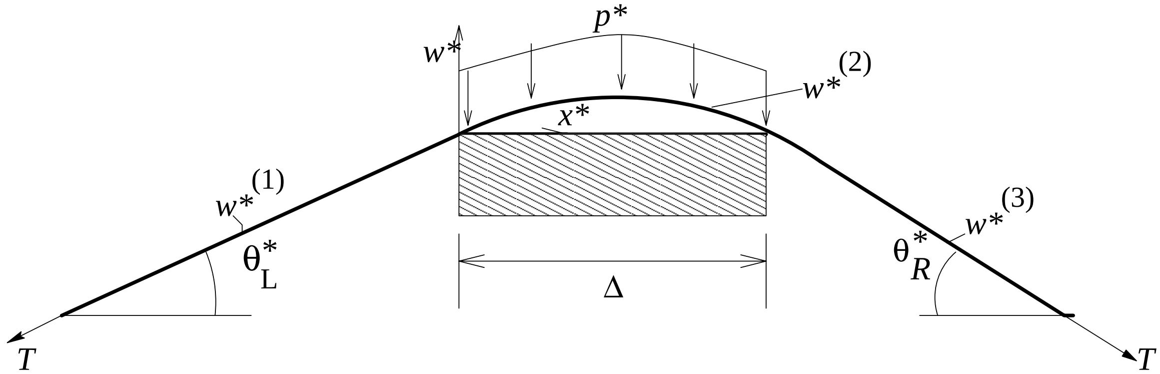


Figure 1 Figure showing a tape wrapped over a flat head under a uniformly distributed negative pressure p^*.

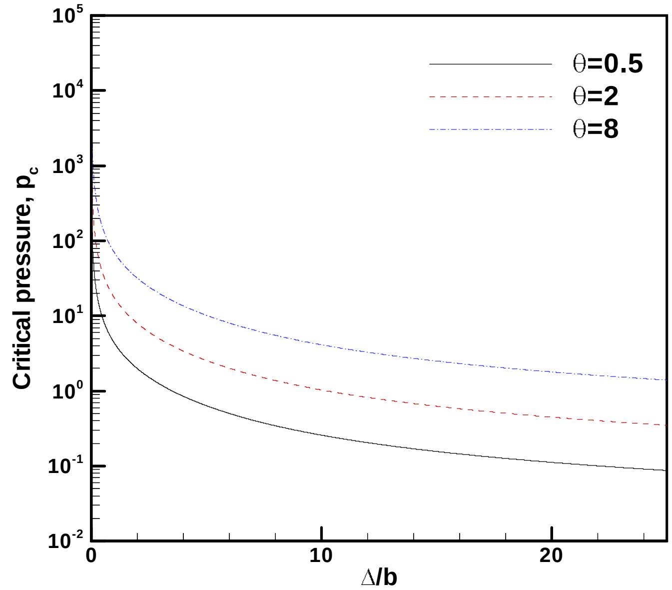


Figure 2 The critical pressure p_{cr} as a function of the Δ/b ratio and the wrap angle θ.

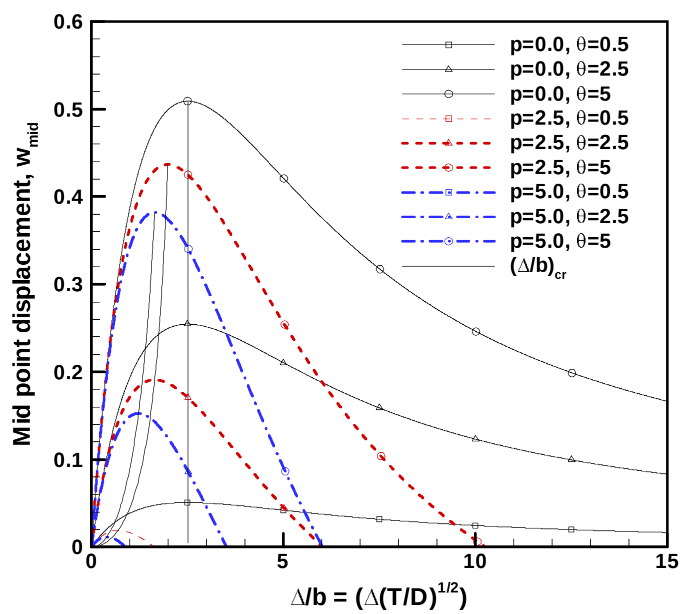
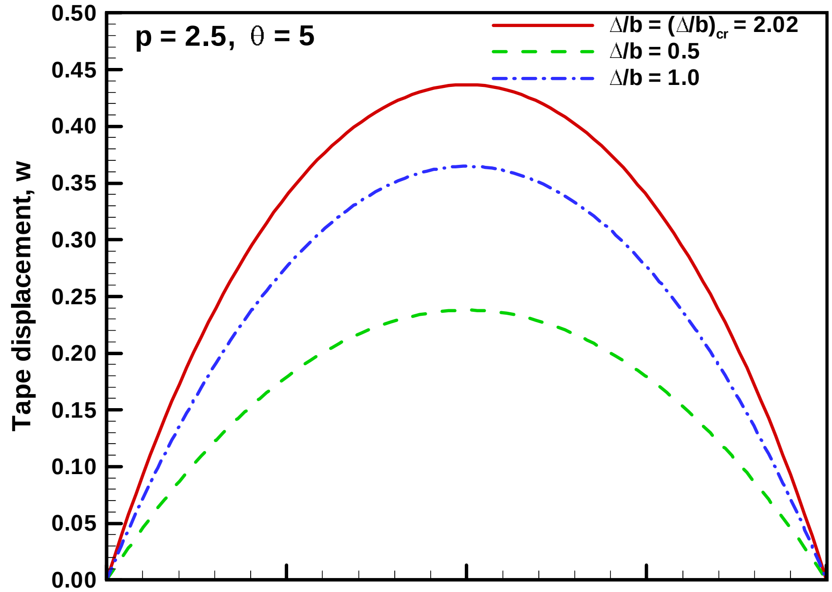
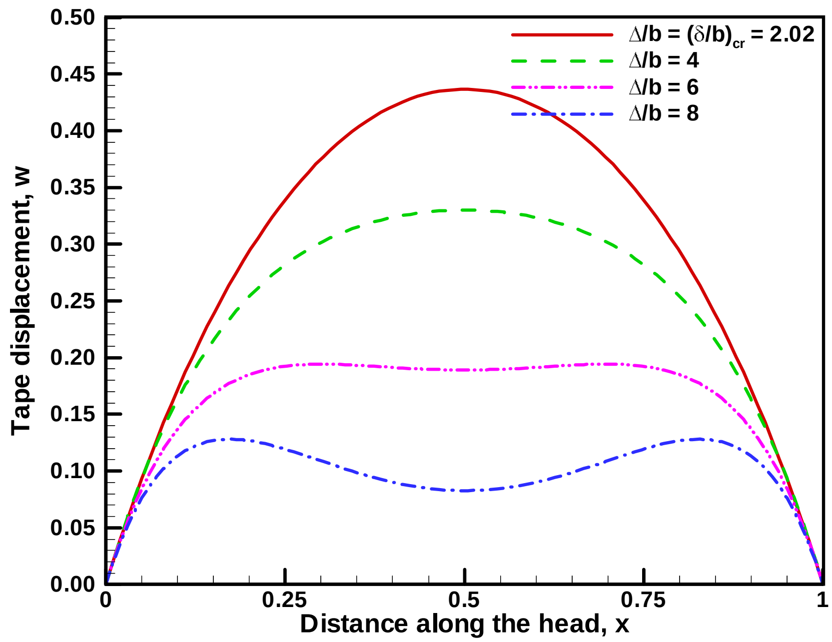


Figure 3 The mid-point tape displacement as a function of the Δ/b ratio for different pressures p and wrap angles θ. Also shown in this figure are the curves ploted by the critical $(\Delta/b)_{cr}$ values for different pressures. The displacement behavior of the tape changes at $(\Delta/b)_{cr}$.



a) Δ/b < (Δ/b)cr case



b) Δ/b > (Δ/b)cr case

Figure 4 The tape displacement profiles for a) $\Delta/b < (\Delta/b)_{cr}$ and b) $\Delta/b > (\Delta/b)_{cr}$ when p = 2.5 and θ = 5. Note that for case in a) w increases with increasing Δ/b and for the case in b) w decreases with increasing Δ/b.

Appendix-1: Deflections of a Non-Tensioned Tape Subject to Negative Pressure

The deflections of a tape wrapped over a flat head without any tension, but under a negative pressure is presented. This case is presented for completeness. In the case where there is no tension in the tape the deflection equation reduces to,

$$\frac{d^4 w}{dx^4} = -\overline{p}\left(H(x) - H(x-1)\right), \tag{12}$$

where p = 12 ($p^*/E)(1-\nu^2)(\Delta/b)^4$ and the other parameters are normalized as before. The boundary conditions of this problem are,

a. On the left and right sides supports of the head, the moments are zero and the displacements are fixed. This is represented as,

$$x = -L: \quad w = -\delta, \quad \text{and} \quad \frac{d^2 w}{dx^2} = 0 \qquad x = 1 + L: \quad w = -\delta, \quad \text{and} \quad \frac{d^2 w}{dx^2} = 0 \tag{13a-d}$$

b. On the edges of the head, the tape displacement is zero and the slope and the curvature of the tape are continuous. These are represented as,

$$\begin{aligned} x^* = 0: \quad & w^{(1)} = w^{(2)} = 0, & x^* = \Delta: \quad & w^{(2)} = w^{(3)} = 0, \\ & \frac{dw^{(1)}}{dx} = \frac{dw^{(2)}}{dx}, & & \frac{dw^{(2)}}{dx} = \frac{dw^{(3)}}{dx}, \\ & \frac{d^2 w^{(1)}}{dx^2} = \frac{d^2 w^{(2)}}{dx^2}, & & \frac{d^2 w^{(1)}}{dx^2} = \frac{d^2 w^{(2)}}{dx^2}. \end{aligned} \tag{14a-h}$$

Then the solution of equation (12) becomes,

$$w_1(x) = \frac{36\delta - L^2 \overline{p}}{12L(3+2L)} x - \frac{24\delta + L\overline{p}}{8L(3+2L)} x^2 - \frac{24\delta + L\overline{p}}{24L^2(3+2L)} x^3, \qquad \text{for } -L \le x \le 0,$$

$$w_2(x) = \frac{36\delta - L^2 \overline{p}}{12L(3+2L)} x - \frac{24\delta + L\overline{p}}{8L(3+2L)} x^2 + \frac{p}{12} x^3 - \frac{p}{24} x^4, \qquad \text{for } 0 \le x \le 1, \tag{15a,b,c}$$

$$\begin{aligned} w_3(x) = & -\frac{24\delta + L(1+L)(1+2L)\overline{p}}{24L^2(3+2L)} + \frac{72\delta(1+L) + L\ (3+2L(3+L))\overline{p}}{24L^2(3+2L)} x - \\ & \frac{(24\delta + L\overline{p})(1+L)}{8L^2(3+2L)} x^2 + \frac{24\delta + L\overline{p}}{24L^2(3+2L)} x^3, \qquad \text{for } 1 \le x \le L. \end{aligned}$$

Note that in this case the exponential terms do not enter the solution; the tape displacements follow a fourth order term over the head and third order terms in the free span. As before, the *dependent variable* of the problem is w and the *independent variable* is x. The *parameters of the problem* are p, L and δ, where $L = L^*/\Delta$ and $\delta = \delta^*/\Delta$ and the dimensional variables L^* and δ^* are the coordinates of the supports measured from the surface of the head.