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To: Millimeter-wave VLBI Group

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Subject: Analytical solution to polarizer loss question.

Simple Treatment

The key to a simple treatment of the polarizer is assuming that we can view the metal vane medium as a homogenous dielectric. We ignore any problems that arise due to fringing fields near the vane edges and idealize the situation as a section of stacked parallel plate waveguides :

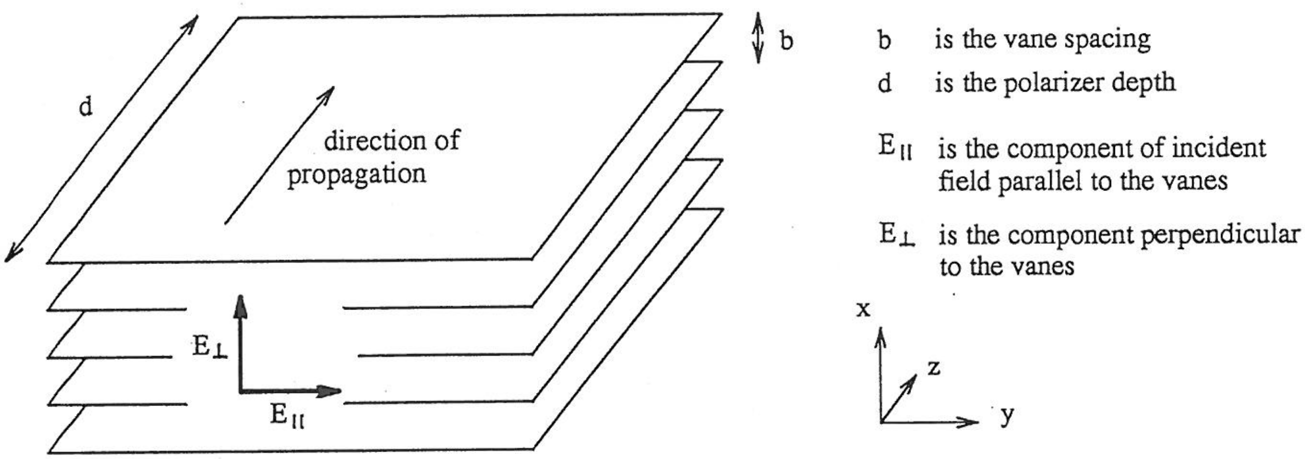

Figure 1: Schematic view of polarizer

Adopting this approach, we can write the electric fields in the polarizer with the following z dependence :

$$E_{\|} \propto e^{ik_{\|}z} \quad with \quad k_{\|} = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{\lambda}{2b}\right)^2} \tag{1}$$

$$E_{\perp} \propto e^{ik_{\perp}z} \quad with \quad k_{\perp} = \frac{2\pi}{\lambda} \tag{2}$$

In other words, the component of incident plane wave polarized in the x direction will pass through the polarizer unaffected. It already satisfies all boundary conditions imposed by the metal plates. The y component, however, will see a waveguide with cutoff wavelength $\lambda_c = 2b$ and effective index of refraction :

$$n_{\|} = \sqrt{1-\left(\frac{\lambda}{2b}\right)^2} \tag{3}$$

The net effect will be to change the relative phase of these orthogonal components which is exactly what we need to build our quarter-wave plate.

A Fabry-Perot type analysis of transmission through the polarizer yields two relations which, when combined, specify the required polarizer depth and vane spacing. Viewing the polarizer as a dielectric slab, we can derive (or find in Born & Wolf) the intensity of the reflected wave:

$$\text{Reflection Intensity} = \frac{4\rho^2 \sin^2\left(\frac{\delta}{2}\right)}{(1-\rho^2)^2 + 4\rho^2 \sin^2\left(\frac{\delta}{2}\right)} \tag{4}$$

$$\text{where} \qquad \delta = \frac{4\pi d n_{||}}{\lambda} \qquad \text{and} \qquad \rho = \left(\frac{1-n_{||}}{1+n_{||}}\right)$$

So, to completely cancel all reflective intensity, we require that $\delta = 2m\pi$ where m is an integer; or, equivalently :

$$d = \frac{m\lambda}{2n_{||}} \tag{5}$$

From the same analysis, we can find the phase difference between the two polarizations of E field and require that it be 90 degrees (for circular polarization) to get the second relation :

$$\frac{\pi}{2} = \frac{2\pi d}{\lambda}(1-n_{||}) - \arctan\left(\frac{\rho^2 \sin(2\delta)}{1-\rho^2\cos(2\delta)}\right) \tag{6}$$

Ultimately, if these two equations are satisfied, our polarizer will advance the phase of $E_{||}$ by 90 degrees and transmit all the energy of the incident wave. When we combine (6) and (5), we find that there is a family of polarizers for a given wavelength :

$$b_m = \frac{\lambda/2}{\sqrt{1-\left(\frac{2m}{2m+1}\right)^2}} \tag{7}$$

$$d_m = \frac{(m\lambda/2)}{\sqrt{1-\left(\frac{\lambda}{2b_m}\right)}} \tag{8}$$

This prescription guarantees us a reflectionless quarter-wave plate at the wavelength λ for each integral value of m. But although it appears that any value of m will do, we have already made an implicit assumption that severely limits the range of m. The general expression for the parallel propagation constant is :

$$k_{||,n} = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{n\lambda}{2b}\right)^2} \tag{9}$$

which means that our value of $k_{||}$ corresponds to the lowest order TE mode in the waveguide. Including higher order terms makes the analysis much more complex so we restrict ourselves to considering only the TE_{10} mode ($n = 1$). To this end we design the waveguide such that all modes with $n > 1$ have pure imaginary propagation constants and are exponentially damped. Specifically we want :

$$\sqrt{1-\left(\frac{\lambda}{2b}\right)^2} \quad \text{to be real}$$

$$\text{and} \quad \sqrt{1-\left(\frac{n\lambda}{2b}\right)^2} \quad \text{to be imaginary if } n > 1.$$

This simplification introduces the restriction $1/2 < b/\lambda < 1$ which forces a limit on how high m can go. The table below shows the acceptable m values for $\lambda = 86\text{GHz}$ and the resulting polarizer dimensions. Theoretical results of the three possible designs are also shown in figures 1 and 2.

m	b_m (mm)	d_m (mm)
1	2.34	2.62
2	2.91	4.36
3	3.39	6.11

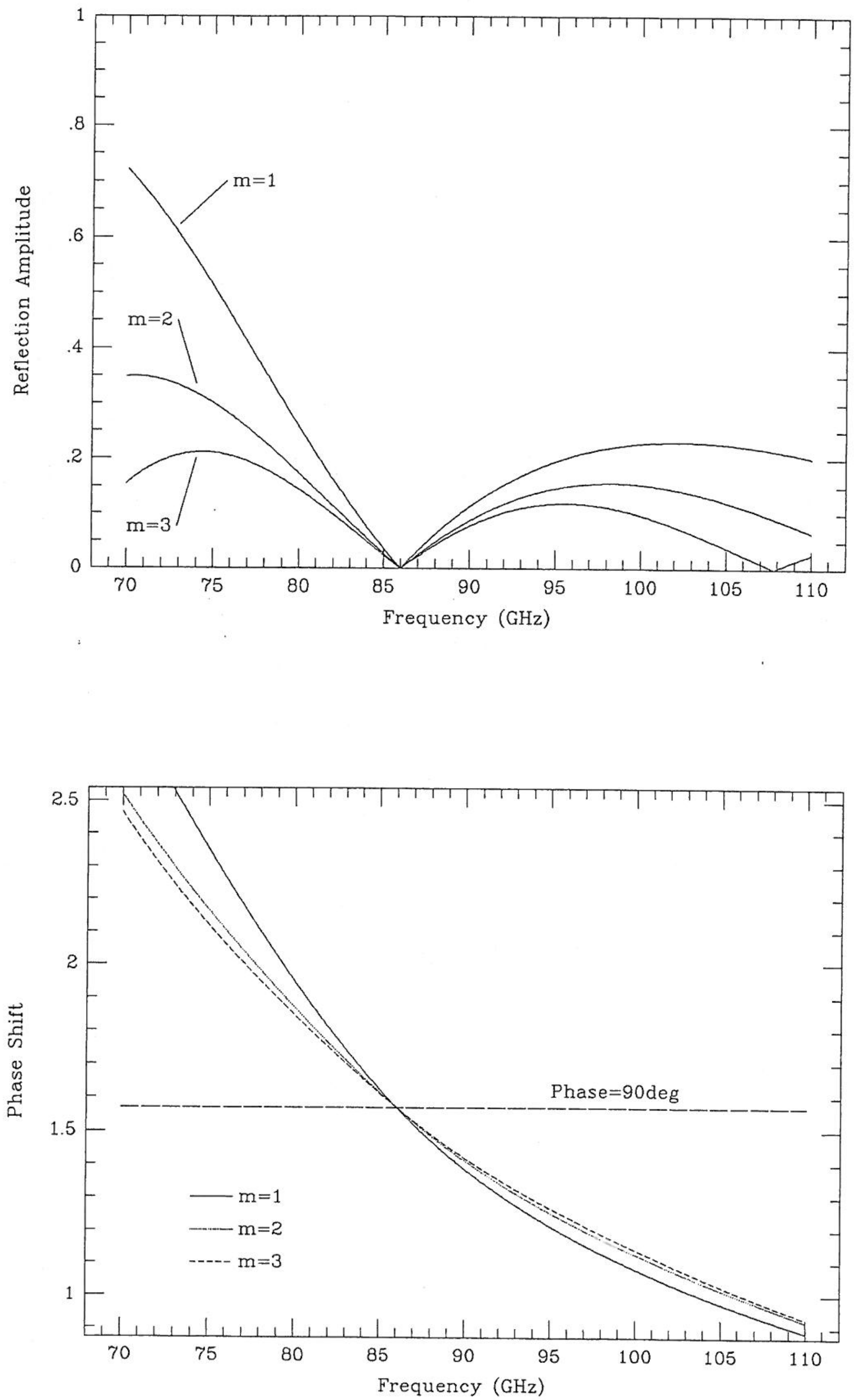

Figure 2: Reflection and phase shift from simple model of polarizer.

Complex Treatment

A more exact look at polarizer performance means addressing the non homogeneity of the metal vane media. In 1947, Carlson and Heins (C+H) were the first to rigorously solve a related problem : the reflection and phase shift of a plane wave incident on an infinite half plane of spaced metal plates. In 1950, Lengyel extended their results to the case of a slab of metal vane media and established a credible agreement with experiment. It should be noted that the final analytical results are expressed in terms of infinite sums, and their evaluation in 1950 was not easy. We've had a much better time of things using computers at Haystack and MIT.

In this section we outline the method of C+H, give Lengyel's results and numerically solve for the E-M fields present in the C+H infinite half-plane problem. Calculation of the fields will enable us to estimate additional losses due to the increased magnetic fields near the vane edges.

C+H formulate the problem as a contour integral over the vane surfaces using Green's theorem. When they impose all boundary conditions, they are left with an integral equation which they have to solve in order to get the surface currents on each vane. Their next step is to fourier transform the integral (which leads to an easier equation) and solve for the current. This step is really quite hard - we have just compressed 10 pages of very complicated math into one line. The final result for the E_y field is in the form of a complex contour integral which is evaluated by looking at the singularities of the integrand. Specifics are not that crucial but physically, each of the singularities corresponds to a TE mode in the waveguide. So, as you might expect, the solution for $E_y(x, z)$ will be a sum of waveguide modes weighted by coefficients determined by C+H's contour integral.

For the case of $f = 86$GHz, $a = 3.2$mm, $d = 5.25$mm the E-M fields just inside the metal vanes are shown in Fig. 3.

In his paper, Lengyel assumes a lossless interface to show how the phase of a wave transmitted through the polarizer is related to the phase shift in the C+H paper. He shows that for a polarizer surrounded by air,

$$\rho' + \rho'' - 2\tau' = \pm\pi \tag{10}$$

where

$$\begin{aligned} \rho' &= \text{phase of reflection from front surface of polarizer} \\ \rho'' &= \text{phase of reflection from back surface of polarizer} \\ \tau' &= \text{phase of transmission across air - metal vane interface} \end{aligned}$$

He uses this relation and the standard reflection/transmission relations for a slab with air on both sides to write :

$$|\text{Reflection Amp.}| = \frac{2\rho|\sin\Psi|}{\sqrt{(1-\rho^2)^2 + 4\rho^2\sin^2\Psi}} \tag{11}$$

$$\text{phase shift} = \rho' + \Psi' + \pi - \frac{2\pi d n_{||}}{\lambda} \tag{12}$$

where

$$\Psi = \rho'' + \frac{2\pi d}{\lambda}$$

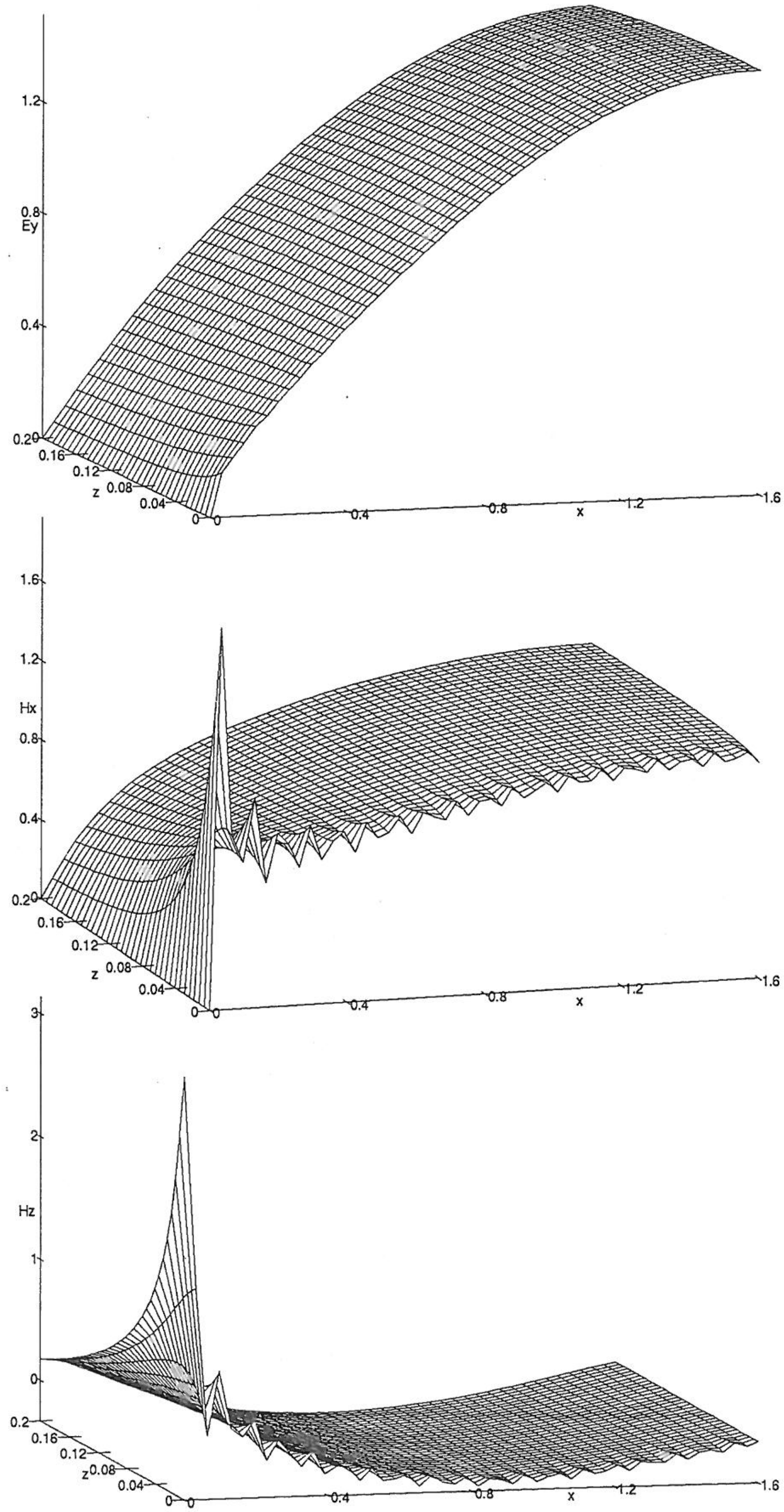

Figure 3: Ey, Hx, Hz Fields just inside the metal vanes. x ranges from vane surface to midway between vanes. Spatial dimensions in mm.

$$\Psi' = \arctan\left\{\left(\frac{1+\rho^2}{1-\rho^2}\right)\tan\Psi\right\}$$

and the expressions for ρ' and ρ'' are

$$\rho'(x) = 2\left\{x(\ln 2 - 1) - \sum_{n=2}^{\infty}(-1)^n\left[\arcsin\left(\frac{x}{n}\right) - \frac{x}{n}\right]\right\} \tag{13}$$

$$\rho''(y) = -2\left\{y(\ln 2 - 1) - \sum_{n=2}^{\infty}(-1)^n\left[\arcsin\left(\frac{y}{\sqrt{n^2-1}}\right) - \frac{y}{n}\right]\right\} + \pi \tag{14}$$

where

$$x = \frac{2b}{\lambda} \text{ and } y = \frac{2bn_{||}}{\lambda} \tag{15}$$

Figure 5 shows graphs comparing Lengyel's predictions with those of the simple approach.

Experiment

In order to asses how well the theory predicted polarizer behavior, we set up a test fixture to measure transmission. Our main goals were to take measurements with the incident E field vector perpendicular, parallel, and also at 45 degrees to the vanes. One expects the first two orientations to pass nearly all incident signal and the last to cut the signal intensity by a factor of 2. The setup is shown here :

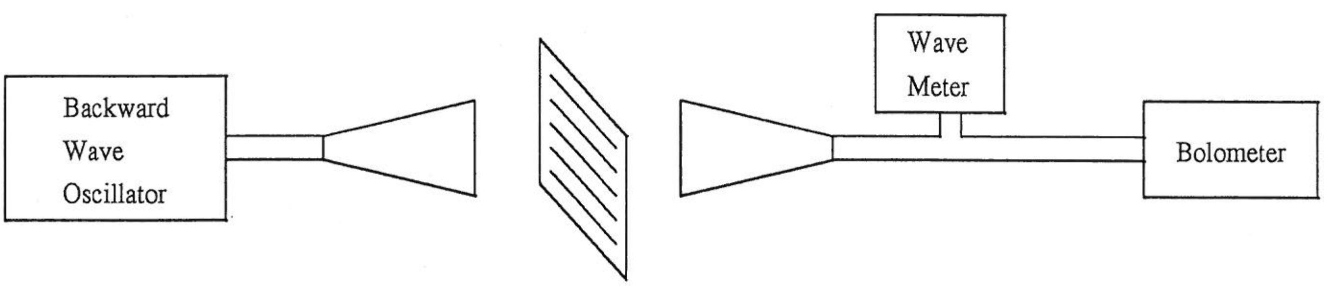

Figure 4: Experiment setup

The aperture of each horn was reduced using echo-sorb plugs to ensure that all path lengths from oscillator to bolometer differed by no more than $\lambda/3$. This was done to reduce the problem of phase front curvature. Another problem involving reflections along the signal path was not so easily dealt with. It turned out that moving the horns together or apart caused the output to oscillate. This was interpreted as fluctuations in signal frequency combining with reflections to cause destructive interference at the detector. An effort was made using a frequency sweep mode of the BWO to average over these unwanted oscillations but results were not as good as expected.

Figure 6 shows transmission for the three orientations. We also graph, for comparison, the expected dependence of 45 degree transmission on phase shift over this frequency range. Each point is an average of a few readings and, based on the variance of these readings, the points shown are good to within 10%.

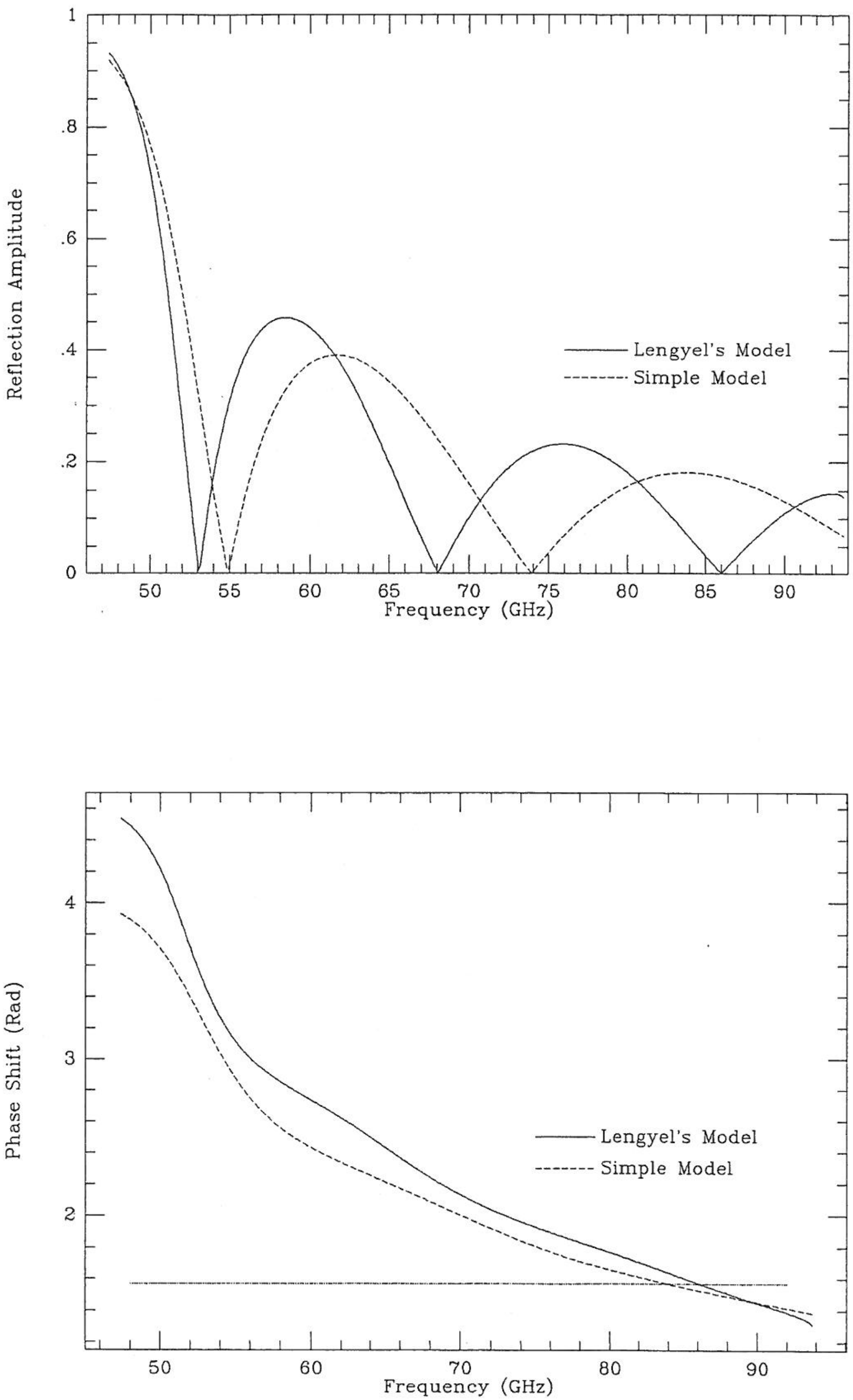

Figure 5: Comparison of Lengyel's reflection and phase shift with those of the simple treatment.

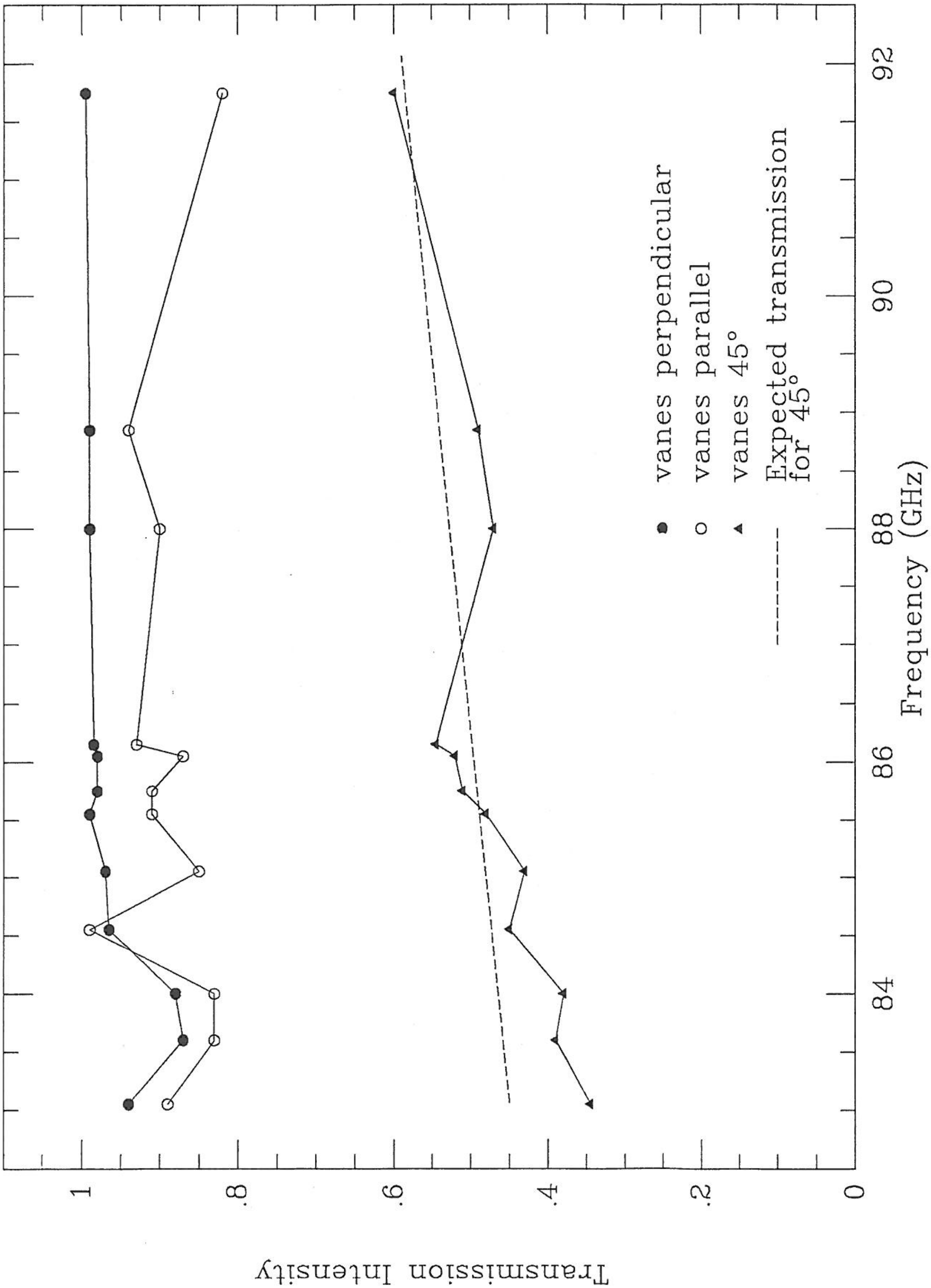

Figure 6: Transmission results for three polarizer orientations. The dashed line is the expected 45 degree transmission.

Losses

As it turns out, the largest contribution to the loss will probably come from the phase error introduced by variations in vane spacing. The prototype polarizer was designed for a spacing of 3.2mm and, when measured, showed an average spacing ($\bar{b}$) of 3.28mm with a standard deviation (σ) of 0.17mm. The polarizer frame was then re-tensioned in a vise to reduce any buckling. When it was re-assembled these numbers changed to $\bar{b} = 3.2$mm and $\sigma = 0.12$mm. To get an idea of how this variation would effect loss, we wrote a phase integral making the vane spacing follow a normal distribution with a specified average and variance. As an approximation, the simple phase model was used. The expression for transmission is :

$$\text{Transmission} = \left| \frac{\int \exp\left(-x^2/2\sigma^2\right) \exp \frac{2\pi i d}{\lambda} \left(\sqrt{1 - \left(\frac{\lambda}{2b}\right)^2} - \sqrt{1 - \left(\frac{\lambda}{2(b-x)}\right)^2} \right) dx}{\int \exp\left(-x^2/2\sigma^2\right) dx} \right|^2 \tag{16}$$

For a $\sigma = 0.15$mm we find a loss of 2.6% and an increase of σ to 0.3mm pushes the loss to 5.7%. A new polarizer design using spring tensioned vanes is being finished and will hopefully reduce this source of loss.

As with any waveguide component, we have to consider ohmic losses due to the finite conductivity of the guiding material. The loss in parallel plate waveguide for the dominant mode is given by :

$$\text{loss} = e^{-2\beta z} \tag{17}$$

$$\beta = \frac{1}{b}\sqrt{\frac{c}{2\sigma b}} \frac{(\lambda/2b)^{\frac{3}{2}}}{\sqrt{1 - \left(\frac{\lambda}{2b}\right)^2}} \tag{18}$$

with σ the conductivity. This amounts to a loss of approximately 0.15% in the polarizer. But we also have to note that the metal vanes distort the normal waveguide fields near the vane edges causing additional loss. In general, the time averaged power absorbed per unit area is roughly :

$$\frac{dP_{\text{loss}}}{da} = \frac{\mu\omega\delta}{4} \left| H_{||} \right|^2 \tag{19}$$

where $H_{||}$ is the tangential magnetic field at the vane surface and δ is the skin depth. Using worst case values of the surface H field yeilds an additional loss of 0.05%. If we double the theoretical loss to account for surface roughness we get a total ohmic loss of ~0.5%. So, as stated in our earlier memo, ohmic losses should account for no more than 1%.

Thickness of the vanes does play a role in transmission but Lengyel's 1950 paper suggests that the effects are not noticeable until the vanes are quite thick. The analytical approach doesn't easily accomodate vane thickness as a parameter but the numerical solution of AEER does. His program shows that our vane thickness (0.003 mils) is a minor loss issue.