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1 November 1993

To: Millimeter-wave VLBI Group

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Subject: Signal to noise ratio of closure phases, bispectrum and weighted triple product

Interferometric phase noise and SNR

The probability distribution of interferometric phase θ due to Gaussian noise is

$$p(\theta) = \frac{e^{-s^2/2}}{2\pi} \int_0^\infty r e^{-r^2/2} e^{sr\cos\theta} dr \tag{1}$$

where s is the SNR (be definition). For large SNR

$$\langle \theta^2 \rangle^{1/2} \approx 1/s \tag{2}$$

Loss factor and the closure phase noise

It is convenient to define a loss factor[1]

$$L(s) = \langle \cos\theta \rangle = \int_{-\pi}^{+\pi} \cos\theta \, p(\theta) \, d\theta \tag{3}$$

$$= s \left[\frac{\pi}{8}\right]^{1/2} e^{-s^2/4} \left[I_o\left(s^2/4\right) + I_1\left(s^2/4\right) \right]$$

$$= \left[\frac{\pi}{8}\right]^{1/2} s \quad \mathit{for}\ s << 1 \tag{4}$$

$$= 1 - \left(1/(2s^2)\right) \quad \mathit{for}\ s >> 1$$

where I_o and I_1 are hyperbolic Bessel functions. For a single sample of closure phase

$$L(s_c) = L(s_1)\, L(s_2)\, L(s_3) \tag{5}$$

where s_1, s_2, and s_3 are the SNRs of the 3 baselines that form a closure triangle and s_c is the SNR of the closure phase. s_c is given by the inverse of the loss function

$$s_c = L^{-1}\left(L(s_1)\,L(s_2)\,L(s_3)\right) \tag{6}$$

and

$$\langle\theta_c^2\rangle^{1/2} = \left[\int_{-\pi}^{+\pi} \theta_c^2\, P(\theta_c)\, d\theta_c\right]^{1/2} \tag{7}$$

where P is the convolution of interferometric phase probability distributions. For the purposes of estimating the rms closure phase the same distribution as given in Equation 1 can be used. At this time this has only been shown to be true by simulations to an accuracy of 1%. (Possibly it can be proved that there is an equality.)

Phase noise of closure phase averages

Closure phases of many data segments can be averaged to improve the SNR. In practice, either the closure phasor or the bispectrum are averaged.

$$A = \frac{1}{N}\sum_i W_i\, e^{i\theta_j} \tag{8}$$

where $W_i = 1$ for phasor averaging and $W_i = (a_1 a_2 a_3)_i$ for the bispectrum

We consider three cases as follows:

Case 1: Low SNR on one baseline

In this case the SNR of the sum S is independent of the segmentation since

$$S_c = L^{-1}\left(L\left(s_1\right)\right) = s_1 \tag{9}$$

and the SNR of each segment decreases with the square root of the segment duration

$$s = s_o\, N^{-1/2} \tag{10}$$

where s_o is the SNR of the full unsegmented data set, and

$$S = sN^{1/2} = s_o \tag{11}$$

when segments are averaged. However, Equation 11 is only valid for the bispectrum. In the case of the phasor average for $s << 1$

$$\langle A \rangle = \frac{1}{N} \sum \langle e^{i\theta} \rangle = \left[\frac{\pi}{8}\right]^{1/2} s \tag{12}$$

$$\langle |A|^2 \rangle = \frac{1}{N^2} \sum \sum \langle e^{i\theta_i} e^{-i\theta_j} \rangle = \frac{1}{N} \tag{13}$$

whereas

$$\langle A \rangle = \frac{1}{N} \sum a_i e^{i\theta_i} = s \tag{14}$$

and

$$\langle |A|^2 \rangle = \frac{1}{N^2} \sum \sum \langle a_i a_j e^{i(\theta_i - \theta_j)} \rangle = 2/N \tag{15}$$

so that the SNR of the phasor average is reduced by a factor of $(\pi/4)^{1/2}$ or about 10% of the SNR is lost.

Case 2: Equal SNR on all three baselines

$$S_c = L^{-1}\left(L^3(s)\right) \approx \left[\frac{\pi}{8}\right] s^3 \qquad \textit{for } s<<1$$

$$\approx \left[\frac{1}{3}\right]^{1/2} s \qquad \textit{for } s>>1$$

$$\textit{and} \quad s \approx s_c N^{1/2}$$

In both Case 2 and 3, the improvement in SNR with $N^{1/2}$ is only approximate and like in case 1 there is a difference in averaging phasors and the bispectrum. Figure 1 shows the SNR and rms closure phase as a function of segment length for Case 2 and 3 for a value of $s_o = 7$. The bispectrum is better than phasor averaging, however, better performance for some ranges of segment SNR can be achieved by making

$$W_i = (a_1 a_2)_i^{1/2} \qquad \textit{for Case 2}$$

$$W_i = (a_1 a_2 a_3)_i^{1/3} \qquad \textit{for Case 3}$$

and we call this the weighted triple product.

The reason for the non-optimal behavior of the bispectrum and phasor averaging is the result of noise in the weight W_i which results in imperfect weighting and is most pronounced for $s<3$. I have only been able to analyze this for Case 1. For cases 2 and 3, I had to resort to computer simulation.

Figure 1 illustrates the relative insensitivity of the SNR to the segmentation as long as the SNR for each segment is greater than unity. It is therefore advantageous to shorten the segments to this point to minimize the signal loss through atmospheric phase fluctuations. An extremely complex analysis of the noise in the closure phase has been made by Kulkarni[2] - this analysis covers additional points using a greatly simplified approach.

The table below shows the performance of the bispectrum and weighted triple product relative to the averaging of phasors for various ranges of SNR on each "weak" baseline.

SNR	N - 1	N = 2 WEIGHTED	N = 2 BISPECTRUM	N = 3 WEIGHTED	N = 3 BISPECTRUM
0.5	14	23	26	10	15
1	14	14	14	19	16
1.5	11	8	5	9	2
2.0	9	6	3	5	-2
2.5	8	5	2	3	-3
3.0	6	3	1	2	-3
3.5	4	2	1	1	-2
4.0	2	1	0	1	-2

Table. Performance of optimal weighted triple product and bispectrum relative to phasor averaging. Performance values are in percent. SNR is the SNR for each weak baseline. N = number of weak baselines.

References

[1] A.E.E. Rogers, A.T. Moffet, D.C. Backer and J.M. Moran, "Coherence limits in VLBI observations at 3-millimeter wavelength", *Radio Science*, Vol. 19, No. 6, pp. 1552-1560, Nov-Dec. 1984.

[2] S.R. Kulkarni, "Self-noise in interferometers: radio and infrared", *A.J.*, Vol. 98, No. 3, pp. 1113-1130, Sept. 1989.

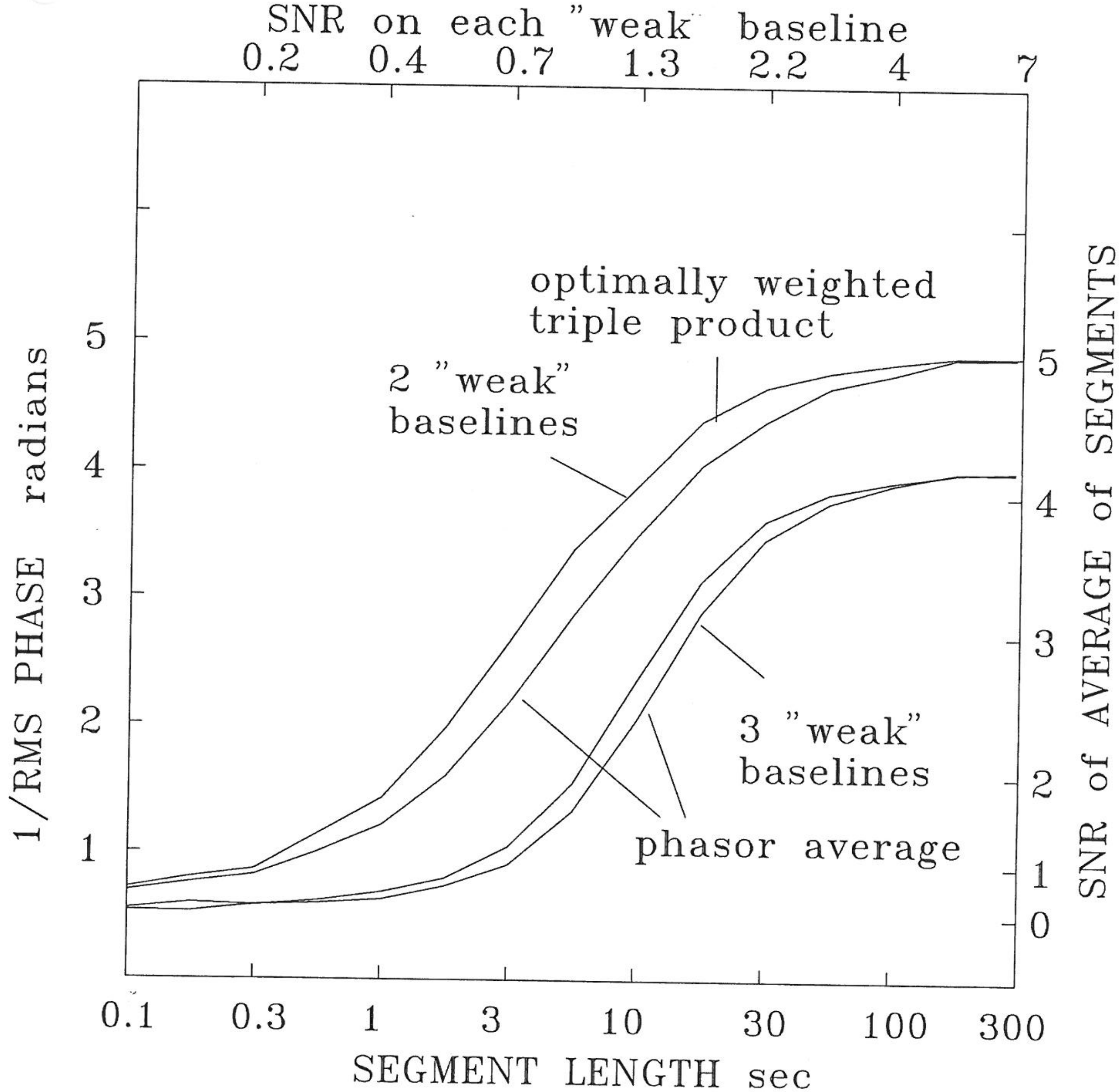

Figure 1. Closure phase noise vs segment length