Covariance Analysis of the Simultaneous Fit of Group Delay and dTEC in *fourfit*

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Rationale

The fringe fit in *fourfit* is performed by finding that group-delay which maximizes the coherent sum of the residual fringe phasors over time and frequency. When the ionosphere is fit, *fourfit* simply searchs over a grid of potential differential ionosphere TEC values and finds a maximum coherent phasor sum via a parabolic interpolation of the gridded values. This method of finding the maximum coherent sum w.r.t. the group-delay has been shown to be equivalent to using least-squares estimation in the region of the maximum.

For that reason we have estimated the errors in the delay and TEC estimates by a covariance analysis using linear least squares.

Phase model

Let us model the observed phase as a function of frequency as follows

$$\phi(f) = \tau_g * (f - f_0) + \phi_0 - 1.3445/f * \delta TEC \tag{1}$$

where:

ϕ phase (rot)

f frequency (GHz)

f_0 reference frequency (GHz)

τ_g group-delay (ns)

δTEC differential TEC (TECU $\equiv 10^{16}/m^2$)

ϕ_0 phase at f_0 (rot)

The constant phase parameter, ϕ_0, might be usefully estimated at a variety of reference frequencies. In the future, if and when the broadband system uses phase-delays referenced to a constant phase at DC, its value may be 0. On the other hand, *fourfit* currently performs a group-delay fit to the slope of phase vs. frequency just over the region of the frequency channels that were employed. Its algorithm, which maximizes the coherent sum of the counter-rotated phasors, has the effect of solving for the mean phase over the sampled frequency channels. The reported standard deviation for the phase is simply $1/snr$ in radians. This is actually an error if the reference frequency is not the mean frequency, as there should also be a term taking into account the uncertainty in the group-delay, multiplied by the difference between the reference frequency (at which the phase is reported) and the mean frequency of the sequence.

Least-Squares Analysis

In weighted linear least-squares the normal matrix, A_{ij}, which is the inverse of the covariance matrix, is defined by

$$A_{ij} = \sum_k \frac{1}{\rho_k^2} \frac{\partial \phi}{\partial \beta_i} \frac{\partial \phi}{\partial \beta_j} \tag{2}$$

The weights, ρ_k, are typically the measurement errors of the dependent variable, which is this case is the phase determined for the k^{th} frequency channel.

The partial derivatives $\frac{\partial \phi}{\partial \beta_i}$ are then given by

$$\frac{\partial \phi}{\partial \beta_0} \equiv \frac{\partial \phi}{\partial \tau_g} = f_k - f_0 \tag{3}$$

$$\frac{\partial \phi}{\partial \beta_1} \equiv \frac{\partial \phi}{\partial \phi_0} = 1 \tag{4}$$

$$\frac{\partial \phi}{\partial \beta_2} \equiv \frac{\partial \phi}{\partial \delta TEC} = \frac{b}{f_k} \tag{5}$$

where we've defined $b \equiv -1.3445$ for convenience. Then the normal matrix A is given by

$$A_{ij} = \sum_k \frac{1}{\rho_k^2} \begin{bmatrix} (f_k - f_0)^2 & f_k - f_0 & b\frac{f_k - f_0}{f_k} \\ f_k - f_0 & 1 & \frac{b}{f_k} \\ b\frac{f_k - f_0}{f_k} & \frac{b}{f_k} & \frac{b^2}{f_k^2} \end{bmatrix} \tag{6}$$

The standard deviations of the parameters, σ_i are given by

$$\sigma_i = \sqrt{A_{ii}^{-1}} \tag{7}$$

and the parameter correlations c_{ij} are

$$c_{ij} = \frac{A_{ij}^{-1}}{\sqrt{A_{ii}^{-1}}\sqrt{A_{jj}^{-1}}} \tag{8}$$

Numerical Results

3480.40	5720.40	6840.40	10680.40
3448.40	5688.40	6808.40	10648.40
3384.40	5624.40	6744.40	10584.40
3320.40	5560.40	6680.40	10520.40
3224.40	5464.40	6584.40	10424.40
3096.40	5336.40	6456.40	10296.40
3064.40	5304.40	6424.40	10264.40
3032.40	5272.40	6392.40	10232.40

Table 1: Broadband frequency sequence (in MHz)

The utility of the broadband frequency sequence (see Table 1) for determining both group-delay and TEC was analyzed using MATLAB, in order to find out parameter correlations and standard deviations. The formulation of the phase model and the least-squares equations are those shown in equations (1) through (8).

Fits were performed both with and without estimating the ionosphere in addition to the group-delay and DC phase, and for a variety of reference frequencies. In order to facilitate comparisons with *fourfit* we used a real scan, which had an snr of 254.6. When there was no ionospheric fit the standard deviation resulting from the least-squares estimate of the group-delay agreed precisely with the current *fourfit* calculation (see equation (9)), which is based upon rms spanned bandwidth and doesn't take into consideration the extra degree of freedom introduced by the ionospheric TEC.

$$\sigma_{mbd} = 1/(2\pi \cdot f_{rms} \cdot snr) \tag{9}$$

Table 2 shows the standard deviation of the fit parameters in both cases - with or without fitting for the ionosphere, and for 3 different values of the reference frequency. Note that the estimate of the group-delay standard deviation went up by about a factor of 2.65 whenever the ionosphere was also estimated. This results from the high correlation between the two parameters, which can be seen (e.g.) in Table 6.

It should also be noted that the standard deviation of the phase, for the case of no ionosphere and the reference frequency at the mean frequency ($f_0 = \bar{f}$) is half of what fourfit finds in this single-sideband example. This is due to the fourfit phase being reported at the edge of the channel, rather than the center. The phase error then depends on both the single-band delay error and the phase error at mid-band. In the case where we have multi-band delays that can be used rather than single-band delays, e.g. when using digital backends with multitone phase cal to adjust for instrumental delays, it is not clear that this factor of 2 is still the correct formulation.

	f_0	τ_g (ps)	ϕ_0 (deg)	δTEC $(TECU)$
no ionosphere	$\bar{f}$	0.239	0.22	–
no ionosphere	6.0	0.239	0.23	–
no ionosphere	0.0	0.239	0.60	–
w/ ionosphere	$\bar{f}$	0.634	1.42	0.016
w/ ionosphere	6.0	0.634	1.52	0.016
w/ ionosphere	0.0	0.634	2.83	0.016

Table 2: Comparison of standard deviations in 2 and 3 parameter fits, for 3 different ref. frequencies

	τ_g	ϕ_0
τ_g	1.000	0.000
ϕ_0	0.000	1.000

Table 3: Correlation matrix without δTEC estimated, $f_0 = \bar{f}$ (mean frequency)

	τ_g	ϕ_0
τ_g	1.000	−0.172
ϕ_0	−0.172	1.000

Table 4: Correlation matrix without δTEC estimated, $f_0 = 6$ GHz (canonical ref. freq)

	τ_g	ϕ_0
τ_g	1.000	−0.927
ϕ_0	−0.927	1.000

Table 5: Correlation matrix without δTEC estimated, $f_0 = 0$ (DC reference freq)

	τ_g	ϕ_0	δTEC
τ_g	1.000	−0.914	−0.926
ϕ_0	−0.914	1.000	0.987
δTEC	−0.926	0.987	1.000

Table 6: Correlation matrix with δTEC estimated, $f_0 = \bar{f}$ (mean frequency)

	τ_g	ϕ_0	δTEC
τ_g	1.000	−0.925	−0.926
ϕ_0	−0.925	1.000	0.989
δTEC	−0.926	0.989	1.000

Table 7: Correlation matrix with δTEC estimated, $f_0 = 6$ GHz (canonical ref. freq)

	τ_g	ϕ_0	δTEC
τ_g	1.000	-0.979	-0.926
ϕ_0	-0.979	1.000	0.977
δTEC	-0.926	0.977	1.000

Table 8: Correlation matrix with δTEC estimated, $f_0 = 0$ (DC reference freq)