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To: Broadband Development Group

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Subject: Notes on phase cal extraction, power levels, total rail count, and SNR

Overview: When setting the phase calibration power level for the broadband development receiver, trade-offs must be made between average phase cal power levels, the number of rails in a given channel, the time interval used to extract phase estimates, and the precision of the extracted phase estimates. In this memo formalisms are developed to define the relationships between these quantities in order that one may quantify the phase calibration amplitude given the rail spacing, processing time, phase cal power level, and required phase estimation precision.

Background and Assumptions:

I only consider signals in each individual DBE channel (i.e. 32 MHz channel) and not the entire downconversion process in the interest of simplicity¹. Furthermore, I assume phase cal extraction is performed in through analog signal processing so I do not consider affects such as two-bit sampling and aliasing. Lastly, I assume that the average power of phase cal within the channel bandwidth is equally distributed among the rails.

1. Phase Calibration Amplitude-Phase Extraction Processing

The phase calibration phases are estimated, basically, by evaluating the Fourier transform of the scan data at the frequency of the phase cal rail undergoing extraction, though there are some subtleties involving normalization of the integrals. Formally put,

the amplitude and phase estimates, \hat{a} and $\hat{\phi}$ respectively are defined as²:

$$\hat{a}(f_o) = \frac{1000}{\sqrt{E}} \sqrt{\frac{F_{cs}^2}{F_{cos}} + \frac{F_{ss}^2}{F_{sin}}} + \frac{1}{2} \tag{1a}$$

¹ The reader should be aware that the downconversion LO and phase cal rail spacing determine exactly how many rails will appear in any given DBE channel since the simplicity of the analysis presented here does not directly address the dependence of these parameters.

² The formulae displayed here were derived from the code `bpcal.c`

$$\hat{\phi}(f_o) = \tan^{-1} 2 \left\{ \frac{-F_{\cos} F_{ss}}{F_{\sin} F_{cs}} \right\} \quad (1b)$$

$$F_{cs} = \int_{T_{pce}} s(t) \cos(2\pi f_o t) dt \quad (1c)$$

$$F_{ss} = \int_{T_{pce}} s(t) \sin(2\pi f_o t) dt \quad (1d)$$

$$F_{\cos} = \int_{T_{pce}} \cos^2(2\pi f_o t) dt \quad (1e)$$

$$F_{\sin} = \int_{T_{pce}} \sin^2(2\pi f_o t) dt \quad (1f)$$

$$E = \int_{T_{pce}} s^2(t) dt \quad (1g)$$

where $s(t)$ represents the data detected in the channel bandwidth, f_o is the baseband frequency at which the phase and amplitude estimates are being extracted, T_{pce} is the time duration used to extract the estimates, and $\tan^{-1} 2$ is the 4-quadrant inverse tangent function.

Simplifications can be applied to (1) using the following relations:

$$\int_{T_{pce}} \cos^2(2\pi f_o t) dt = \int_{T_{pce}} \sin^2(2\pi f_o t) dt = \frac{T_{pce}}{2} \quad \text{for } 2\pi f_o T_{pce} \gg 1 \quad (2a)$$

$$\int_{T_{pce}} s(t) \cos(2\pi f_o t) dt = \frac{T_{pce}}{2} \hat{a}_r(f_o) \cos(\hat{\phi}(f_o)) \quad (2b)$$

$$\int_{T_{pce}} s(t) \sin(2\pi f_o t) dt = \frac{T_{pce}}{2} \hat{a}_r(f_o) \sin(\hat{\phi}(f_o)) \quad (2c)$$

where, \hat{a}_r is an estimate of the peak-voltage amplitude of the phase cal rail at f_o as determined by the complex Fourier transform of $s(t)$. The condition ' $2\pi f_o T_{pce} \gg 1$ ' in (2a) is valid for the typical phase cal extraction scenario (e.g. $f_o = 16 \text{ MHz}$, $T_{pce} = 10 \text{ s}$). In (2b) and (2c), I recognize the real and imaginary parts of the Fourier transform of $s(t)$ at f_o are related through the phase estimate of the phase cal rail.

Substituting the relations made in (2) into the formulae in (1) and simplifying terms, the amplitude and phase estimates of the phase cal tone at f_o are given by:

$$\hat{\phi}(f_o) = \tan^{-1} 2 \left\{ \frac{-F_{ss}}{F_{cs}} \right\} \quad (3a)$$

$$\hat{a}(f_o) = 1000 \sqrt{\frac{T_{pce}}{2E}} \hat{a}_r(f_o) + \frac{1}{2} \quad (3b)$$

In observing (3a), one can see that the phase estimate of the phase cal tone is nothing more than the phase of the Fourier sample of $s(t)$ at f_o . In observing (3b), the amplitude of a phase cal tone is a normalized value (correlation coefficient?) representing the rail's energy relative to the total energy ' E ' contained in the signal $s(t)$ ³.

Equation (1g) defines the total energy contained in the receiver signal and in order to complete the processing formalism, ' E ' must be related to the average phase cal power and the receiver noise. The total signal, $s(t)$, is a sum of the phase cal signal and the receiver noise signal:

$$s(t) = \sum_{k=0}^{N_r-1} a_r(f_{LO} + k\Delta f) e^{2\pi(f_N + k\Delta f)t} + s_m(t) \quad (4)$$

where, $s_m(t)$ is the receiver noise defined as a Gaussian white noise voltage signal having an rms voltage σ_r , N_r is the number of rails in the channel bandwidth, f_{LO} is the LO frequency which relates the rail's equivalent sky frequency to its corresponding baseband frequency, k is the tone number, a_r is the voltage amplitude of the rail at the specified frequency, and t is time within the phase cal processing interval. It should also be noted that f_o coincides with one of the rail frequencies such that for a given tone number k , $f_o = f_{LO} + k\Delta f$

Since the phase cal tones are uncorrelated with one another and are also uncorrelated with the receiver noise, we need only consider the sum of the squares of each term in (11) when calculating the total signal energy. That is:

$$E = \int_{T_{pce}} s^2(t) dt = T_{pce} \left(\sum_{k=0}^{N_r-1} \frac{a_r^2(f_N + k\Delta f)}{2} + \sigma_r^2 \right) \quad (5)$$

³ In bpcal.c, the phase cal amplitude as defined in (3b), is actually scaled by a factor of 1000

Further simplification can be made by recognizing that the average power contained in the phase calibration signal is given by one-half the sum of the squares of each tone's peak-amplitude:

$$\bar{P}_{pc} = \sum_{k=0}^{N_r-1} \frac{a_r^2(f_N + k\Delta f)}{2} \quad (6)$$

Equation (5) relates the total energy contained in $s(t)$ in terms of the average receiver noise and phase cal power. With this relation a more specific expression for the phase cal amplitude can be developed by substituting (5) and (6) into (3b):

$$\hat{a}(f_o) = 1000 \sqrt{\frac{1}{2(\bar{P}_{pc} + \sigma_r^2)}} \hat{a}_r(f_o) + \frac{1}{2} \quad (7)$$

Single Rail Power SNR:

The receiver's noise power is given by⁴:

$$\sigma_r^2 = kT_r B_c \quad (8)$$

where T_r is the receiver's noise temperature and B_c is the bandwidth of the DBE channel filter (32 MHz). The time-average phase cal power injected into the front-end of the receiver is:

$$\bar{P}_{pc} = \eta \sigma_r^2 \quad (9)$$

where η is the parameter dictating the strength of the time-average phase cal power (all rails in the 32 MHz channel) relative to the receiver noise.

Assuming the phase cal power is equally distributed across all rails contained in B_c , the time-average power contained in a single rail is:

$$\bar{P}_{pcr} = \frac{\bar{P}_{pc}}{N_r} = \frac{\eta k T_r B_c}{N_r} \quad (10)$$

The phase cal extraction zooms in on the spectral region where the phase cal rail exists and reduces the observable bandwidth of the signal centered at f_o to B_{pce} as dictated by the time duration used to extract the rail's phase:

$$B_{pce} = \frac{1}{T_{pce}} \quad (11)$$

The noise power available after phase cal extraction is:

⁴ Receiver gain factors have been suppressed as the following analysis only considers the relative difference in power between the phase cal tones and the receiver noise; gain factors are common to both.

$$\sigma_{rpce}^2 = \frac{kT_r}{T_{pce}} \quad (12)$$

The power signal-to-noise ratio of a single phase cal rail after extraction is:

$$SNR = \frac{\bar{P}_{pcr}}{\sigma_{rpce}^2} = \eta \frac{T_{pce} B_c}{N_r} \quad (13)$$

Precision of Phase Estimates:

The presence of receiver noise in $s(n)$ limits the precision of phase estimates. The strength of the phase cal rail relative to the available noise in the estimate is defined by (13). Assuming the $SNR \gg 1$ (so we have Gaussian statistics), the uncertainty in the phase estimate based on the SNR is given simply by the following:

$$\hat{\sigma}_\phi = \frac{1}{\sqrt{SNR}} \quad (14)$$

In practice, if we require the 95% confidence interval ($2\sigma_\phi$) of the phase estimates to be 1° or less, this necessitates that the single rail SNR be greater than 13053 or 41.2 dB.

Specification of Phase Calibration Amplitude for Desired Phase Precision

The phase cal amplitude can be expressed solely in terms of η , and N_r . In order demonstrate this relationship the peak-voltage amplitude of the rail must be related to it's time-average power:

$$\bar{P}_{pcr}(f_o) = \frac{a_r^2(f_o)}{2} \quad (15)$$

Then substituting equations (9), (10) and (15) into (7)⁵, combining terms, and canceling factors common to the numerator and denominator, reveals the dependence of the phase cal amplitude on the parameter η - the ratio of the time-average phase cal power in the channel relative to the noise power in the same channel):

$$\hat{a} = 1000 \sqrt{\frac{\eta}{\eta+1} \frac{1}{N_r} + \frac{1}{2}} \quad (16)$$

⁵ Since the actual rail voltage amplitude a_r expressed in (15) and the estimated rail voltage \hat{a}_r in (7) are not technically the same quantity, in general, this substitution is void because we do not have apriori knowledge of a_r . However, in the case when $SNR \gg 1$ (i.e. when a high degree of phase precision is required) the actual quantity and the estimated quantity converge and the substitution is justified.

Given the relationship of the SNR on the experiment parameters to, η , T_{pce} , B_c , and N_r (equation (13)), and the relation of the SNR to the phase cal phase precision, ϕ_σ (equation (14)), the following describes the dependence of the phase estimate precision on the experiment parameters

$$\phi_\sigma = \sqrt{\frac{N_r}{\eta T_{pce} B_c}} \quad (17)$$

Finally, equations (16) and (17) can be used to relate the required phase cal amplitude, a , to the desired phase estimate precision, ϕ_σ , through the experiment parameters, T_{pce} , B_c , and N_r as follows:

$$a = 1000 \sqrt{\frac{1}{\phi_\sigma^2 T_{pce} B_c + N_r}} + \frac{1}{2} \quad (18)$$

There are various pragmatic considerations that must be examined when choosing η , N_r and T_{pce} ⁶, however, these trade-offs will not be examined here. Once the desired η , N_r have been determined, these quantities can be substituted into (19) which will yield the required phase cal amplitude given the original phase precision requirement.

Comparison of Measured and Calculated Phase Calibration Amplitude

To test the validity of the equation (16), a simple experiment was conducted with a microwave noise generator, phase calibration signal generator, updown converter (Luff = 7645.1 MHz, Nyquist zone 2 output), and digital backend (DBE1); the experimental setup is shown in Figure 1. The 7th polyphase frequency channel spanning 781.4 to 813.4 MHz IF was arbitrarily chosen for this experiment. In this experiment, the phase cal power level was set to approximately 0.1% (it is purely coincidental that the attenuator in figure 1 is 29 dB) of the noise power in the 32 MHz channel. Given the UDC Luff frequency, the 8880 MHz phase cal tone is translated to an IF of 799.6 MHz and baseband frequency of -16.4 MHz; this tone was also arbitrarily targeted for the experiment.

A spectrum analyzer was used to measure the total analog IF power in the aforementioned polyphase filter channel as well as the power contained in the 799.6 MHz phase cal tone. The total IF power in the 32 MHz channel as measured at the IF2 monitor output (derived from DBE1 input with 15dB coupler) of the DBE1 was $\sigma_r^2 = 100 \mu W$ (-40 dBm) while the power measured in the 799.6 MHz tone was $\bar{P}_{pcr} = 70.795 nW$ (-71.5 dBm). Using the Mark5B+ recorder, a test scan of the superposed noise and phase calibration signals was made, 20ms of which was subsequently processed using `bpca1`. The phase cal amplitude as reported by `bpca1` was 25 (amplitude is unitless per (18)).

⁶ As an example of a practical consideration, it is desirable to minimize η in order to reduce noise introduced into station-to-station cross-correlations by the phase cal signal. In contrast, η must be large enough to be detected with the specified SNR within the detection period. Also, the power in the rail must overcome the power in a supposed spurious signal occupying the same frequency as the rail under extraction.

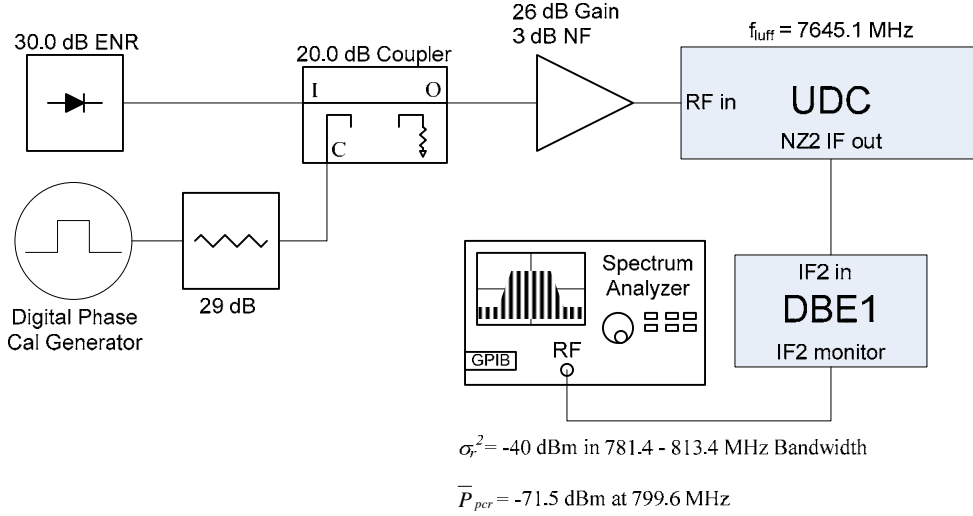


Figure 1: Test setup used to compare measured and calculated phase calibration amplitudes

In the case when $\eta \ll 1$, equation (16) collapses to the following form using (9) and (10):

$$\hat{a} = 1000 \sqrt{\frac{\eta}{N_r}} + \frac{1}{2} = 1000 \sqrt{\frac{\bar{P}_{pcr}}{\sigma_r^2}} + \frac{1}{2} \quad (19)$$

Given the measured σ_r^2 and \bar{P}_{pcr} , the calculated phase cal amplitude is $\hat{a} = 27$, which is in reasonable agreement with measured value of 25.

Observations

For the relatively small phase cal power levels (i.e. $\eta \ll 1$) that are encountered in practice, the result presented in the previous section indicates that the continuous phase cal amplitude analysis presented here is sufficient for the cases expected to be encountered at healthy VLBI stations. As mentioned in the background section, the analysis conducted here was done so assuming continuous mathematics and I expect that the 2-bit quantization of the data as recorded by the Mark5B+ will impart some deviation from the continuous case when $\eta \ll 1$ is no longer valid. Since the analytically analyzing phase cal extraction of 2-bit quantized samples becomes unwieldy, such analysis is probably best left to numerical simulation (i.e. Monte Carlo analysis).