To: Holographers
From: Brian Corey
Subject: Holography resolution functions

This pedagogical note was motivated by Alan Rogers's 19 March 1992 memo on the panel structure that is unresolved by 91×91 holography. The purpose is to present the analytical forms for the resolution functions and their corresponding frequency response functions. Some approximations are made (e.g., I use continuous Fourier transforms here, whereas in practice FFT's are used), but the effects on the results are negligible. All integrals extend from −∞ to +∞.

Before transformation to the map plane, the complex beam data are first gridded onto the rectangular (u, v) grid to form the data set B(u, v), and then a window function W(u, v) is applied to the data. u and v are the direction cosines u = sin θ cos φ and v = sin θ sin φ, where θ and φ are the polar and azimuthal angles about the center of the beam.

The process of transforming to the map plane may be expressed as

\[
M(x, y) = \iint B(u, v) W(u, v) e^{ik(ux+vy)} \, du \, dv
\] (1)

where \(M(x, y)\) is the complex map in the (x, y) aperture plane, and \(k = 2\pi/\lambda\). In the usual manner, eq. (1) may also be written as the 2D convolution of two functions:

\[
M(x, y) = \frac{k^2}{4\pi^2} \iint M_0(x - \xi, y - \eta) R(\xi, \eta) \, d\xi \, d\eta
\] (2)

where \(M_0(x, y)\) is the map obtained from eq. (1) with \(W(u, v) = 1\) everywhere (i.e., it is the map with the highest possible resolution, for which the beam pattern has been measured over the full sky), and \(R(x, y)\) is the resolution function given by

\[
R(x, y) = \iint W(u, v) e^{ik(ux+vy)} \, du \, dv
\] (3)

We have used two different window functions in the analysis of holography data. The original square window has the form

\[
W_s(u, v) = \begin{cases} 
1 & \text{for } |u| \leq u_{\max}, \, |v| \leq u_{\max} \\
0 & \text{elsewhere}
\end{cases}
\]
More recently the standard has become the circular function

\[ W_c(u, v) = \begin{cases} 
1 & \text{for } \sqrt{u^2 + v^2} \leq u_{\text{max}} \\
0 & \text{elsewhere}
\end{cases} \]

For a \((2N + 1) \times (2N + 1)\) map, \(u_{\text{max}}\) is \(N \Delta u\), where \(\Delta u\) is the spacing between grid points in the \((u, v)\) plane. All Haystack maps have had \(\Delta u = \sin 0.032^\circ = 0.00056\). For a 91\times91 map, \(u_{\text{max}} = 0.0251\).

The corresponding resolution functions in the map plane, obtained from eq. (3) and normalized to unity at the origin, are

\[ R_s(x, y) = \frac{\sin kxu_{\text{max}} \sin kyu_{\text{max}}}{kxu_{\text{max}} kyu_{\text{max}}} \]  
\[ R_c(x, y) = 2 \frac{J_1(kru_{\text{max}})}{kru_{\text{max}}} \]  

where \(r = \sqrt{x^2 + y^2}\), and \(J_1\) is the Bessel function of the first kind and first order. Cross-sections through these functions for a 91\times91 map are plotted in Figure 1. The resolution function represents the "impulse response" of the system, in the sense that a delta-function spike in the high-resolution map \(M_0(x, y)\) gets smeared out by the windowing process to look like \(R(x, y)\).

A complementary way to look at the effect of the window function is to compute the frequency response function \(S(k_x, k_y)\), which is the inverse Fourier transform of the resolution function:

\[ S(k_x, k_y) = \frac{k^2}{4\pi^2} \iint R(x, y) e^{-i(k_xx + k_yy)} \, dx \, dy \]  

The spectral content of the map is simply \(S(k_x, k_y)\) times the spectral content of the unwindowed map \(M_0(x, y)\). By eq. (3) [or you can do the integrals, as a check of eqs. (4) and (5)],

\[ S(k_x, k_y) = W(k_x/k, k_y/k) \]  

For instance, the frequency response function for the circular window is

\[ S(k_x, k_y) = \begin{cases} 
1 & \text{for } \sqrt{k_x^2 + k_y^2} \leq ku_{\text{max}} \\
0 & \text{elsewhere}
\end{cases} \]

For \(u_{\text{max}} = 0.0251\) and \(\lambda = 2.54\) cm, \(ku_{\text{max}} = 2\pi / (101\) cm). Hence, a 91\times91 map produced with a circular window will exhibit unattenuated all power on wavelength scales > 100 cm across the aperture, and will have no power on shorter scales.

One caveat: Strictly speaking, the resolution function and frequency response function do not apply to the map phase, since the phase is a nonlinear function of the map \(M(x, y)\), and Fourier transformation is a linear process. In the case of small \((\ll 1\) radian) map phases and large-scale-only \((> \lambda/u_{\text{max}})\) amplitude variations, however, the resolution function may be applied directly to the map phases. For 91\times91 maps, these two conditions hold reasonably well over most of the Haystack aperture plane, except at the edge of the dish and near the quadrapod legs and subreflector.
Figure 1. Cross-section through origin, parallel to $z$-axis, of normalized resolution functions for $91 \times 91$ holography maps, with $u_{\text{max}} = 0.251$ and $\lambda = 2.54$ cm. (1) Circular window resolution function $R_c(x,y)$, which is circularly symmetric. (2) Square window resolution function $R_s(x,y)$. 